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PCK Tools

Measures of Central Tendency: Student Misconceptions and Strategies for Teaching

The study of statistics is important in the elementary and middle grades because society frequently organizes and expresses data numerically (statistically). From early childhood we are exposed to statistics in weather predictions, test grades, sports, poll data, charts and graphs, and newspaper ads. We are inundated with statistical information and we must learn how to process this information accurately and effectively to function as knowledgeable citizens in society (Hatfield, Edwards, Bitter, & Morrow, 2005).

One of the first concepts of statistics encountered in daily life is that of "average." The teacher says, "Your *average* score for the spelling tests is 72." The doctor says, "Your weight is below the *average* for an 8-year-old girl." Thus, a child hears the word *average* even before she/he understands its meaning or knows how to calculate it.

Definitions

Average is a description of what is typical in a data set. In other words, it is a measure of "central tendency" of the data. Three measures are commonly used to describe the central tendency of a set of data: mean, median, and mode. The mean is often used interchangeably with average, but average can be used more broadly to include all three measures of central tendency. In other words, average does not merely reference the mean.

Mean: The *mean* or *arithmetic mean* is most commonly used to describe the average of a set of data. The mean is computed by dividing the sum of the numbers in the data set by the number of members in the set.

Median: Median of a set of numbers is the middle member of the set when the members are arranged in the increasing or decreasing order of their value.

Mode: The mode of a set of numbers is defined as the most frequently occurring member in the set.

State Standards

Both the New Jersey and the Texas state standards for mathematics (see Appendix) are similar to the National Council of Teachers of Mathematics *Standards* (2000) in concepts related to measures of central tendency. In sixth grade students are expected to be able to use the median and mode as well as the mean to describe data. By the time they complete eighth grade, they should be able to select and use appropriate measures of central tendency (mean, median, and mode) to describe a set of data for a particular purpose.

Students' Understanding of the Concept of Average

Most of the research literature about middle-school students' understanding of measures of central tendency focuses on the arithmetic mean. Of the 15 studies examined in this review, only four referenced measures other than the mean, such as the median or mode. Two of these studies (Watson & Moritz, 2000; Mokros & Russel, 1995) explored students' understanding of "average" in a broader sense, but did not focus on the mathematical properties of the median or mode. A third (Zawojewski & Shaughnessy, 2000) reported that "middle school students have some difficulty finding the mean and the median ... [and] even greater problems selecting and using different statistics appropriately" (p. 436). Specifically, the authors reported "confusion" about the meaning of the median among eighth graders, and stated that when asked to find the center of a data set, students tend to choose mean over median, regardless of the distribution of data. A fourth study (McGatha, et al., 1998) presented students with word problems and distributions of data and asked them to make decisions based on the data in which mean and median were compared. Each of these studies is discussed further below.

Mokros and Russel (1995) presented students with a series of increasingly complex problems related to average, asking them to solve them and to explain their strategies for doing so. Their analysis of student responses revealed the five conceptualizations of average listed in Table 1 below. It is important to note that, in general, a goal of instruction should be to get youth to a point where they understand average by the fifth conceptualization. While the first four are extremely important in building a concrete understanding of average, the fifth is a more abstract and generalizable understanding.

Table 1. Student Conceptualizations of Average

Concept	Characteristics of Students With This Approach
Average as a MODE	o Lack of flexibility in choosing strategies o Are unable to build a distribution when not allowed to use the given average as a data point o Use the algorithm for finding the mean infrequently or incorrectly o View the mode only as the *most*, not as a representative of the whole data set o Frequently use egocentric reasoning in their solutions
Average as an ALGORITHM	o View finding the average as carrying out a procedure o Often exhibit a variety of useless and circular strategies that confuse total, average, and data o Have limited strategies for determining the reasonableness of their solutions
Average as REASONABLE	o View an average as a representative of the data, both from a mathematical perspective and from a common sense perspective o Use their real-life experiences to judge if an average is reasonable o May use the algorithm for finding the mean o Believe that the mean of a particular data set is not one precise mathematical value, but is an approximation
Average as MIDPOINT	o View an average as a tool for making sense of data o Choose an average that is representative of data, both from a mathematical perspective and from a common sense perspective o Look for a middle to represent a set of data o Use symmetry when constructing a data distribution around the average, but have significant trouble constructing or interpreting nonsymmetrical distributions o Use the mean fluently as a way to check answers
Average as a MATHEMATICAL POINT OF BALANCE	o View an average as a tool for making sense of data o Take into account the values of all the data points o Use the mean for the beginning understanding of the quantitative relationships among data, total, average; they are able to work from a given average to data, from a given average to total, from a given total to data o Break problems into smaller parts and find ‘‘submeans” as a way to solve more difficult averaging problems

Watson and Moritz (2000) conducted a longitudinal study of the developments of students' concepts of average to further investigate Mokros and Russel's frame work for students' conceptions of average and they identified six levels of understanding of average among students in grades 3, 6, and 9. (See Table 2). Their study, together with the works of Mokros and Russel, offers a conceptual framework for teaching the concept of the mean and for analyzing students' misconceptions of average.

Table 2. Levels of Understanding of Average

Levels of Understanding	Nature of Understanding/Proficiency
1. Pre-average	Student is familiar with the term average but does not know its meaning.
2. Single colloquial usage	Student is familiar with the term, and has a general, nonmathematical sense of its meaning (e.g., average means normal).
3. Multiple structures associated with the mean, median, or mode	Student employs multiple definitions of average, some colloquial (normal, mediocre, median construct), common (modal construct), some mathematical (aware of the add-and-divide algorithm).
4. Recognition of the average as a representation of the data set	Student understands that the average is a representation of a data set, and can make inferences about the data based on this understanding.
5. Able to use average to either a) work backward from a decimal mean value or b) find a weighted mean	Student understands the relationship between the mean and data set but cannot consistently or flexibly apply this knowledge to solve complex problems.
6. Able to successfully complete both tasks	Student understands the relationship between the mean and data set, and is able to apply this knowledge to solve complex problems.

Among middle-grade (6-9) students, the researchers found that 43% of sixth graders, 50% of seventh graders, 33% of eighth graders, and 21% of ninth graders demonstrated understanding at Level 3; while 14%, 44%, 33% and 25% of sixth through ninth graders (respectively) were at Level 4. While these findings were based on small sample sizes, the data suggest that most middle-grade students 1) are aware of multiple colloquial meanings of average associated with mean (the add-and-divide algorithm), median (mediocre or middle), or mode (common); and 2) can use the algorithm to find the mean, but do not necessarily know what the mean represents or its relationship to the data. Not surprisingly, students' level of understanding of the concept of average improves by with instruction. With respect to conceptual understandings of average, the researchers found that nearly all middle-school students understand average as the mean, as evidenced by their recitation of the algorithm. In addition, about half of the same students also understood the average as median. Interestingly, seventh graders were more likely to employ at least two definitions of average than either eighth or ninth graders.

The process through which children come to understand the concept of average remains poorly understood. Investigating whether there was some type of developmental pathway for students learning the concept, Strauss and Bichler (1988) identified seven properties of "the average" that are fundamental and tap three aspects—statistical (properties 1-3), abstract (properties 4 and 5), and representative (properties 6 and 7)—of the concept.

1. The average is located between the extreme values.
2. The sum of the deviations from the average is zero.

3. The average is influenced by values other than the average.
4. The average does not necessarily equal one of the values that were summed.
5. The average can be a fraction that has no counterpart in reality (e.g., the average family has 2.3 children).
6. When one calculates the average, a value of zero—if it appears—must be taken into account.
7. The average value is representative of the values that were averaged.

Properties 2, 6, and 7 are much more difficult for students to understand than the rest, and most people do not have more than a superficial understanding of property 7. Not surprisingly, the authors found that among a sample of children ages 8-14, older children understood more of these properties than younger children. Again, this should be attributed to experience and learning with central tendencies in and out of the classroom.

Common Difficulties in Understanding 'the Average'

Students in the middle grades make a number of computational errors when solving problems related to the average, most of which are rooted in a lack of conceptual understanding of the relationship between average and the data that it represents. There are two levels of misunderstanding. At the most basic level, students do not understand the concept of data itself; they can calculate a mean but do not necessarily associate the numbers they add and divide with any type of real-world occurrence.

In order to grasp the concept of average, students first need to understand the idea of data as a way of describing something (e.g., the number of books each student in the class brings to school each day). Once students grasp the concept of data, the next level of comprehension is getting them to see an average as a way of describing that data. Implicit in this understanding is comprehension of the relationship between 1) the individual data points and the data set, and 2) the data set and the average. Students who know the algorithm for calculating the mean often do not understand that the mean represents the data set as a whole, and that the data set is composed of individual data points. If a data point changes or new data points are added, the data set changes, and thus the average may change (adding two 5s to a set that already has a mean of 5 would result in the same mean). This lack of conceptual understanding is most apparent in students' struggles with the mean, but it is equally true with median and modal constructions of average.

Students who associate individual data points with a data set, and in turn understand the average as a way of describing that data set, are far more likely to be able to solve computational problems related to the mean, to compare means, and to compare different measures of central tendency. *Unfortunately, many students are taught the algorithm for finding the average around the same time they are taught division, long before they understand the average as a representation of a set of data*. As a result, students know how to "add and divide," but their application of the algorithm is mechanistic and inflexible (Meyer & Browning, 1995; Carpenter, et al., 1981). This lack of understanding results in five common obstacles when solving problems related to the average:

- **Students are unable to work the algorithm for the mean backwards**. Given the average of a data set and asked to fill in missing values, students are unable to identify them. For example, Cai (1995) analyzed sixth graders' response to the following problem:

  > *Angela is selling hats for the mathematics club. She sold 9 hats in week one, 3 hats in week 2, and 6 hats in week 3. How many hats must Angela sell in week 4 so that the average number of hats sold is 7?*

  Of the 50% of students who responded incorrectly, four types of errors were found. The first two errors come from a misapplication of the algorithm (conceptualizing average as an algorithm rather than a mathematical concept), while the second two errors stem from not understanding that the average takes account of all the data points (not understanding average as a mathematical point of balance).

  1. The students added the number of hats sold in the first three weeks with the desired average (7) and divided by 4.
  2. The students added the number of hats sold in the first three weeks and divided by 3.
  3. The students followed the same process as in #2, then added 3 to the sum and divided by 3, bringing the mean to 7 but using an incorrect divisor.
  4. The students followed the same process as in #2, then gave an answer of 1, the difference between the mean of the first three weeks (6) and the desired mean (7).

  In each case, students failed to multiply the intended average (7) by the correct number of weeks (4) to arrive at the desired total.

- **Students are unable to calculate a weighted mean.** Students often fail to understand the difference between the mean for a data set and a mean based on a set of submeans (for instance, students can be given the means for two sets of data, and the goal is to find the mean of a new set that includes both of these now subsets). For example, Polltasek, et al., (1981) reported that only 40% of college psychology majors were able to solve the following problem:

  > *A student attended college A for two semesters and earned a 3.2 Grade Point Average (GPA). The same student attended college B for three semesters and earned a 3.8 GPA. What is the student's GPA for all his college work?*

  Among the majority of students who answered incorrectly, the most common response was 3.5—the mean of the two submeans presented. While Polltasek and colleagues studied college students, subsequent research has demonstrated the prevalence of this type of error among middle-school students as well (Watson &

Moritz, 2000). Averaging the averages is almost always problematic; it is only appropriate when each average has the same weight.

- **Students are unable to construct a data set around a given mean**. As discussed above, students struggle to grasp the idea that an average is a representation of the values contained in a data set. Students who struggle with this concept are unable to construct a set of possible values around a given average. For example, Watson and Moritz (2000) presented students with a problem in which they are asked to create a graphic representation of the price of nine different brands of potato chips in such a way that the average price was $1.38, with the stipulation that they could not use the mean as a value.

- **Students are unable to use the concept of average to make inferences**. Presented with a set of data points (e.g., temperatures over the course of a week), students are unable to make inferences about the data set, instead relying on preconceived or anecdotal constructions of average (most often modal in young children, and varied in middle-school students) to generalize about the problem. Students see finding the average as the end goal of a task, not looking beyond the algorithm to thinking about what kinds of information this number represents. For example, if students are asked to find the average temperature for a given week, students may add up and divide by 7, thinking this is the point of finding averages (math for math's sake). They rarely think about what the average means, or how it can be used (i.e., to make predictions).

- **Students are unable to select which measure of central tendency is most appropriate for a given problem context.** Students therefore cannot select the appropriate statistic for a given distribution of data. Zawojewski and Shaughnessy (2000) found that when asked to select the appropriate measure of central tendency, students did not use the mathematical characteristics of each to make their decision. Instead, they tended to default to the mean, because it was somehow viewed as more precise than the median, possibly because computing it required that all the values in the data set be included, as opposed to finding a single, central value.

Teaching Strategies to Address These Difficulties

Strategies for teaching average are organized around helping students to build a conceptual understanding of the relationship between data points, a data set, and the average as a representation of the data. With specific reference to the mean, they also emphasize alternatives to the computational algorithm that help students to see the relationship between the mean and the data. Following are six approaches to teaching average:

1. **Deemphasize the computational algorithm**. Helping students to develop a conceptual understanding of average often requires them to move away from a formal procedure for calculating the mean. Cai, et al., (1995) recommend using alternative approaches (specifically leveling, described below) before introducing the algorithm for finding the mean, thus allowing students to develop a

contextualized understanding of the average concept prior to using the algorithm. "Children wedded to the algorithm must be pulled away from their narrow view of average as a procedure to focus on describing and comparing data sets" (Mokros & Russel, 1995, p. 37).

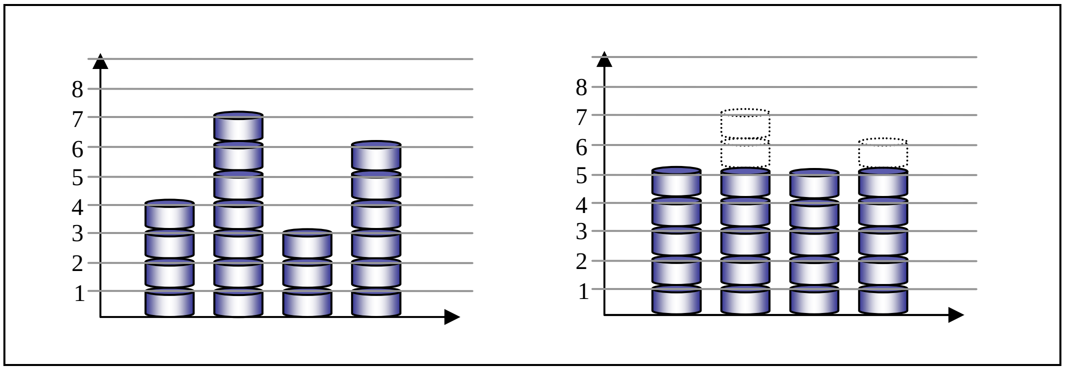
2. **Use data/content that is relevant /tangible for students**. If students appear to view average as a mathematical operation unconnected to a set of data, one option is to begin with a data set that occurs naturally in their daily lives and ask them to make sense of the data as a whole. A second approach suggests introducing the concept by asking students for everyday examples of the use of average, such as the cost of some item, school attendance, or GPA. A third strategy uses familiar and real-world items, such as raisins or M&Ms, as a context to work with and think about concepts of central tendencies (Cai, et al., 1997).

3. **Use relevant/tangible data sets to solve contextual problems**. McGatha, et al., (1998) constructed problems involving students determining in which month they should take a class trip, based on daily and weekly average temperatures in the destination city in previous years. In another problem, students were asked to choose between two basketball players for a tournament based on the number of points scored in previous games. In both of these problems, students were presented with a group of numbers arrayed around a familiar context (points scored, daily temperature) and asked to make statements about the set of numbers as a whole. This recommendation is in line with research suggesting that students develop greater comprehension of the conceptual properties of the mean when they are presented in narrative rather than numerical format (Leon, et al., 1993). Another advantage of asking students to make sense of sets of data as described above is that properly constructed, these types of problems force them to weigh the relative advantages of various measures of central tendency as predictive values. For instance, in the weather problem described above, some students argued for one month over the other based on the *consistency* of temperature, even though the mean temperature for the other month was higher. In another version of this approach (McClain, et al., 2003), students were asked to evaluate the effectiveness of a speed trap by analyzing the distribution (presented graphically) of recorded speeds before and after the trap was implemented.

4. **Leveling/evening out**. Used with graphic representation, this approach asks students to view data points as stacks, or columns, then asks them to "level" the columns by taking away from taller ones and adding to smaller ones. The line at which the stacks or columns are leveled is the mean. Cai, et al., (1997) found that leveling, in conjunction with word problems, significantly improved students'

ability to calculate the mean, and their ability to explain the process for doing so. At the same time, the authors warned that in isolation, leveling can become its own kind of computational algorithm, and must be contextualized within the data to be conceptually effective. Blocks or tiles can also be used with this approach (Post-It notes on a large graph work well, too). In Japanese, Taiwanese, and Chinese textbooks, they use containers of fluids (e.g., water). This approach allows for continuous data while the use of blocks or tiles is limited to discrete data. Moreover, often times the mean is not an integer in which case it is impossible to level the cubes precisely, but this average can be found with fluids. George (1995) found that students who were instructed using a leveling method had a better conceptual understanding of the mean than those who merely learned the computational algorithm.

Figure 1. An example of leveling/evening out



5. Other suggestions:

- Ask students to create data sets of particular properties (e.g., five data points, mean = 6 but 6 is not an element of the data set; six data points, mean = 6, median = 7).
- Use multiple representations. For example, have them create a line plot to depict the distribution for each step in the leveling out process.
- Provide data sets for which the mean is clearly not the best measure of central tendency (e.g., seven 100s and one 0 in the data set).
- Have students consider how the mean changes with the introduction of additional data points (such as zero, the mean, values greater than the mean, etc.)

Appendix

New Jersey Core Curriculum Content Standards

Building upon knowledge and skills gained in preceding grades, by the end of ***Grade 6***, students will:

A. Data Analysis

1. Collect, generate, organize, and display data.

    - Data generated from surveys

2. Read, interpret, select, construct, analyze, generate questions about and draw inferences from displays of data
    - Bar graph, line graph, circle graph, table histogram
    - Range, median, mean
    - Calculators and computers used to process information

3. Respond to questions about data, generate their own questions and hypothesis, and formulate strategies for answering their questions and testing hypothesis.

Building upon knowledge and skills gained in preceding grades, by the end of ***Grade 7***, students will:

A. Data Analysis

1. Select and use appropriate representations for sets of data, and measures of central tendency (mean, median, and mode).

    - Type of display most appropriate for given data
    - Box and whisker plot
    - Calculators and computers used to process information

2. Make inferences and formulate and evaluate arguments based on displays and analysis of data.

Building upon knowledge and skills gained in preceding grades, by the end of ***Grade 8,*** students will:

A. Data Analysis

1. Select and use appropriate representations for sets of data, and measures of central tendency (mean, median, and mode).

    - Type of display most appropriate for given data
    - Box and whisker plot, upper quartile, lower quartile
    - Scatter plot
    - Calculators and computers used to process information
    - Finding the median and mean (weighted average) using frequency data
    - Effect of additional data on measures of central tendency

2. Make inferences and formulate and evaluate arguments based on displays and analysis of data.

3. Estimate lines of best fit and use them to interpolate within the range of data.

4. Use surveys and sampling techniques to generate data and draw conclusions about large group.

Texas Essential Knowledge and Skills Standards

Grade 6

Probability and statistics. The student uses statistical representations to analyze data.

The student is expected to:

(A) select and use an appropriate representation for presenting and displaying different graphical representations of the same data including line plot, line graph, bar graph, and stem and leaf plot;

(B) identify mean (using concrete objects and pictorial models), median, mode, and range of a set of data;

(C) sketch circle graphs to display data; and

(D) solve problems by collecting, organizing, displaying, and interpreting data.

Grade 7

Probability and statistics. The student understands that the way a set of data is displayed influences its interpretation.

The student is expected to:

(A) select and use an appropriate representation for presenting and displaying relationships among collected data, including line plot, line graph, bar graph, stem and leaf plot, circle graph, and Venn diagrams, and justify the selection; and

(B) make inferences and convincing arguments based on an analysis of given or collected data.

Probability and statistics. The student uses measures of central tendency and range to describe a set of data.

The student is expected to:

(A) describe a set of data using mean, median, mode, and range; and

(B) choose among mean, median, mode, or range to describe a set of data and justify the choice for a particular situation.

Grade 8

Probability and statistics. The student uses statistical procedures to describe data.

The student is expected to:

(A) select the appropriate measure of central tendency or range to describe a set of data and justify the choice for a particular situation;

(B) draw conclusions and make predictions by analyzing trends in scatterplots; and

(C) select and use an appropriate representation for presenting and displaying relationships among collected data, including line plots, line graphs, stem and leaf plots, circle graphs, bar graphs, box and whisker plots, histograms, and Venn diagrams, with and without the use of technology.

Probability and statistics. The student evaluates predictions and conclusions based on statistical data.

The student is expected to:

(A) evaluate methods of sampling to determine validity of an inference made from a set of data; and

(B) recognize misuses of graphical or numerical information and evaluate predictions and conclusions based on data analysis.

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