



PCK Tools

Graphs: Student Misconceptions and Strategies for Teaching

The National Council of Teachers of Mathematics (NCTM, 1989) has called for an increased emphasis in the middle grades on (a) reasoning from graphs, and (b) the description and representation of relationships with tables, graphs, and rules. NCTM has also called for an increased emphasis in high school on (a) the connections between problems, functions, and graphs, and (b) the use of computer-based graphing utilities. These increased emphases on understanding and reasoning from graphs are accompanied by a decreased emphasis on the mechanics of plotting graphs with paper and pencil. This may be seen as a natural evolution based on available technology, and more generally, fitting with an educational philosophy emphasizing understanding over rote processes. Despite the availability of tools such as computers and graphing calculators, understanding and using graphs remains a source of difficulty for many students.

This paper summarizes some of the research into students' difficulties understanding graphs and the connections between graphs, data sets, functions, and real-world situations. Please note that there is a *close connection between graphs and functions*, and this is a rich source of student misconceptions and errors. In this paper, these difficulties will be explored from the point of view of graphs, while another paper will be devoted specifically to functions, variables, and symbolic representation.

Research on Graphs

Graphing Situations

Graphs appear in the mathematics curriculum primarily in two ways. First, graphs may be used in the context of measurement and statistics for the presentation and analysis of collections of data. Many types of graphs can be used to represent data sets (such as bar graphs, line graphs, pie charts, etc., described below). Second, graphs may be used in the context of algebra for the visual representation of functions; these are usually line graphs in the middle grades.

When used for statistics, graphs provide students many ways to organize, analyze, and present data. Software tools can simplify some of the mechanical aspects of graph construction and manipulation. These tools facilitate understanding by allowing students to spend more time interpreting graphs in just the ways called for by NCTM (1989). In

addition, students will have more time to explore different types of graphs, helping them to develop a stronger understanding through connecting data with various displays of that data.

In algebraic situations, graphs and equations of functions play mirror roles—equations are the symbolic representation of a relationship, while graphs are the visual representation of that relationship. Traditionally, student tasks in this area involve creating graphs from functions, picking out features of graphs, and solving equations based on their graphs. Research suggests that many students have difficulty connecting graphs to functions. Graphs and functions may also appear in the context of real-world situations, such as the motion of a vehicle or the volume of water in a bathtub. In these instances, students may need to relate to a graph, not only to the symbolic representation of a function, but also to events in a real-world context.

In both cases, it is important to point out that different types of graphical displays are appropriate in different situations. Students not only need to understand how to construct and interpret these graphs, but understand what each makes visible and which is most appropriate for a given situation.

Types of Graphs for Statistics

According to Curcio (1989):

Graphs provide a means for communicating and classifying data. Graphs allow for comparison of data and the display of mathematical relationships that often cannot easily be recognized in numerical form. The traditional, most common forms of graphs found in newspapers, magazines, and advertisements are *picture graphs*, *bar graphs*, *line graphs*, and *circle graphs*. ... More recently, some new plotting techniques have been recommended for inclusion in the [elementary and middle school] curriculum. ... These techniques include line plots, stem-and-leaf plots, and box plots (p. 1).

- **Picture graphs** use uniform icons or pictures to depict a quantity of something. For example, in a picture graph showing gross national product, dollar signs might be used to indicate billions of dollars.
- **Bar graphs** use discrete horizontal or vertical bars to depict quantity. With respect to the GNP example, in a bar graph the height or length of bars represent total GNP, with specific numerical value indicated by the axes or labels on the bars themselves. A histogram is a form of bar graph, related yet distinct. In a histogram, the width (or height) of the bars represents a range of data instead of a single, discrete point. As such, histograms are a way of reducing data by organizing it into intervals.

- **Line graphs** display continuous data, such as longitudinal or functional relationships. For example, a line graph might depict change in GNP over time, with the total value of GNP on one axis and time period on the other. On the other hand, a line graph could also be used to show the relationship between GNP and unemployment, with unemployment plotted on one axis and GNP on the other.
- **Circle graphs**, or pie charts, show the relationships between the parts of a data set and the whole. For example, a circle graph could show what percentage of GNP comes from the health care industry relative to the total.
- **Line plots** depict the occurrence of specific values within a data set, with values depicted horizontally and one X stacked above a specific value for each time that value occurs in the data set. For example, if 12 students take a math quiz and score between 79 and 92, a line plots would include the numbers 79, 80, 81 ... 92 along the horizontal axis. If four students scored 85 on the quiz, four X's would appear above the number 85 on the horizontal axis.
- **Stem and leaf plots** serve a function similar to histograms, but are somewhat simpler to read and construct, as students can see the individual values even as they are represented graphically. In a stem and leaf plot, numerical data are organized into intervals (e.g., 10s or 100s), with the interval making up the “stem” and the specific values comprising the “leaves.” To use the quiz example above, a stem and leaf plot of student scores might include three “stem” values (70, 80, and 90) arranged along the vertical axis. Each score falling within each range would be depicted alongside the stem as a single-digit number. So if four students scored 85 on the quiz, four 5s would appear next to the 80 stem value, along with any other scores in the 80s.
- **Box plots** depict the range of values within a data set, and are a useful tool for understanding statistical distribution. The parameters of a box plot are defined by the minimum and maximum values in the data set (79 and 92 in the quiz example). Horizontal lines link these minimum and maximum values to a box depicting the middle two quartiles (25th to 75th percentiles) of observations. A line bisecting the box, though not necessarily into equal halves, indicates the median value. Thus, looking at a box plot quickly reveals the range, median, and quartiles of a data set.

Students' Understandings and Misconceptions of Graphs

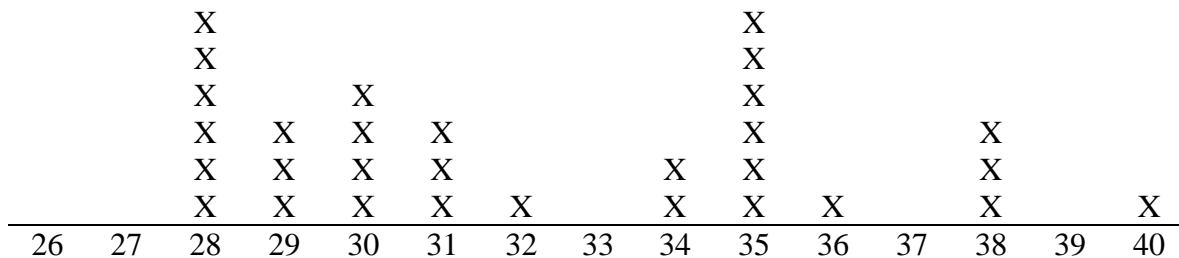
Graph Sense

Factors associated with graph comprehension include prior knowledge of mathematics, the topic of the graph (i.e., what the data represent), and graphical form, such as the relationship between column height and quantity (Curcio, 1987).

Friel, Bright, and Curcio (1997) coined the term “graph sense” to describe students’ capacity to make sense of graphs and the data they represent. Summarizing a wide range of literature on graph comprehension, the authors identified three levels of graph sense:

1. **Reading the data.** Students can answer literal questions about the specific data pictured in a graph. Reading the data requires that students understand what a graph is: What the axes represent, where the data are located, and what the data represent. The authors offer the example of a line plot (see Figure 1), with the x - axis showing the number of raisins in a single box (ranging from 26 to 40). The number of Xs stacked over each number indicates the number of boxes containing that number of raisins. For this graph, a student capable of reading the data would be able to answer a question such as: “How many boxes had 30 raisins in them?”

Figure 1: Line Plot Showing Number of Boxes Containing Specific Quantities of Raisins



2. **Reading between the data.** Students can identify and explain relationships in the data portrayed in the graph. This requires that students understand not only the basic structure of the graph (described above), but also make sense of the variation in values or types of data portrayed. Using the raisin example provided above, a student capable of reading between the data would be able to answer questions such as: *Are there the same number of raisins in each box? How can you tell?*
3. **Reading beyond the data.** Students can extrapolate from the data and relationships portrayed in the graph. This requires that students understand the structure of the graph and the relationships contained therein, but also the context in which the data are presented. In other words, students must be able to use the information presented in the graph to answer questions that extend beyond the data the graph contains. Returning to the raisin example, a student capable of reading beyond the data would be able to answer a question such as: *If you were to open one more box of raisins, how many would you expect to find?* This specific question asks students to interpret the data contained in the line plot, using them to identify some measure of the center of the data set in order to predict an outcome.

These three levels of graph sense have been explored by a variety of researchers (Bertin, 1983; Curcio, 1987; McKnight, 1990; Wainer, 1980, 1992; Carswell, 1992). While specific terminology varies, each employed a construct that involved an elementary understanding of graphs based on simple *extraction* of the data from the graph, an intermediate level of understanding based on *identifying relationships* in the data, and an overall level of

understanding based on moving beyond the data contained in the graph to make broader judgments (Friel, Curcio, & Bright, 2001).

It may be inferred from this literature that students' level of graph sense may be identified by asking specific types of questions in a particular sequence, such as the questions about the raisin line plot described above:

- Entry level questions should focus on the literal content of the graph: the specific value of a data point or the meaning of the axes.
- Intermediate questions should address relationships between data points: changes over time or comparisons between elements (such as the height of bars in a bar graph or the size of sections in a circle graph).
- Overall questions should require predictions or inference based on the data presented.

While it may be possible to assess students' level of "graphicacy," Berg and Smith (1994) caution that the methodology and instruments used by researchers (and teachers) directly influence the outcomes observed. Specifically, the authors found that when presented with graphical interpretation questions in multiple-choice format, students answered significantly more questions incorrectly than if they were asked to construct a graph themselves and then explain their reasons for doing so. This suggests that many questions involving the construction or interpretation of graphs (and the data they represent) often have more than one correct answer, and that multiple-choice instruments do not adequately explicate students' conceptual understandings of the graph and the data it represents. To that end, the authors recommend teaching and assessment approaches (such as free-response questions) that emphasize students' explanation of their choices in constructing or interpreting graphs.

Student Misconceptions of Graphs

Researchers have noted that students have trouble making connections between graphs and other representations, whether those other representations are, for example, data sets, algebraic functions, real-world events, or other types of graphs (Dunham & Osborne, 1991; Knuth, 2000; Dugdale, 1993). Others have noted student errors and misconceptions related to specific types of graphs, and these are noted below.

- **Seeing graphs as pictures.** Friel and Bright (1996) note that in interpreting line plots, students sometimes confuse the number of Xs (or referents) associated with a particular value on the X axis with the value itself. In other words, to return to the raisin example shown in Figure 1, students committing this type of error would think that the number of Xs appearing above the X axis was equal to the total number of raisins in a box, as opposed to reflecting a distribution of the number of raisins in several different boxes.

Along similar lines, several researchers (Kerslake, 1977; Schultz, et al., 1986; Mokros & Tinker, 1987) have found that students have a tendency to think of a graph as a “picture” of the data rather than a representation of it. For example, in a graph showing a movement, students often confused a sloped line showing distance over time with the actual direction of movement, mistaking acceleration for movement in a northwesterly direction (Berg & Smith, 1994). This suggests that students fail to make the conceptual connection between the image of the graph and the data it represents.

- **Line graphs and functions.** Several studies have identified line graphs as the most difficult for students to construct and interpret (Thomas, 1933; McDonald-Ross, 1977). Padilla, McKenzie, and Shaw (1986) identified subskills associated with constructing and interpreting line graphs. To construct a line graph, students need to be able to draw and scale axes, assign “manipulated and responding variables” to each axis, plot points, and use a line of best fit. To interpret a line graph, they must be able to determine the X and Y coordinates of a point, interpolate and extrapolate, state relationships between the variables, and interrelate the results of multiple graphs. In assessing middle- and high-school students’ performance on each of these subskills, the authors found that middle-school students were most successful in plotting points and determining the X and Y coordinates of a point, but struggled with scaling axes and using a best fit line. The authors suggested that these difficulties may represent a lack of the mathematical knowledge required to develop these skills.

Dunham and Osborne (1991) identified a number of common errors in student comprehension of graphs associated with functions, some of which may be generalized to graphs as a whole. First, they discovered that in looking at “points” on the graph of a function, students often fail to recognize them as ordered pairs—“projections” from the X and Y axes. In other words, the value of a point within a function is not associated with its relative position along the axes in question. Second, students fail to understand that the “line” in a graph representing a function is actually made up of an infinite number of ordered pairs produced by that function. The authors speculate that because students learn to graph linear functions using the “slope-intercept” method (calculating the slope and the points at which the line intercepts that axes), students tend to view the line as connecting two or more fixed points. Finally, the authors noted that in reading graphs, students struggled with the concept of scale. Specifically, students could not explain how a change in the scale of one of the axes would affect the shape of the graph. Again, this suggests that students do not associate the shape of a graph (in this case, a function) with the data the graph was intended to represent as indicated by the labels and axes. At the middle school level or below, this confusion may reflect students’ lack of understanding of ratio and proportion—concepts central to understanding scale.

Knuth (2000) studied student understanding of the connection between functions and graphs. Given a question about a graph and related function, students tended to

do complex calculations with the function in order to answer the question, rather than read the answer off the graph. He suggests that students have a limited understanding of the relationship between graphs and functions. He notes that while students often create graphs from functions, they rarely get practice creating functions from graphs.

Another issue that students often struggle with is the notion of linearity. For example, a student plotting a linear function may plot one point incorrectly and connect all of the plotted points with straight line segments that go from point to point. They do not realize, in looking at the graph as a whole, that it displays a nonlinear function. As a second example, students given a graph of a linear function in which the axes are unlabeled and for which they are asked to complete a table of values that could be represented by the function will often pick points that are not linear (e.g., (1,1), (2,2) (3,5), (6,6)).

- **Histograms and bar graphs.** With respect to constructing and interpreting histograms, students struggle with the data reduction required to construct the intervals comprising the individual bars, or columns, within the graph. Specifically, students often failed to construct intervals that were discrete. For example, in creating a histogram depicting the distribution of prices of a given item, students constructed intervals that included \$5.00-\$5.50 and \$5.50-\$6.00, creating overlap between the two intervals. The authors speculated that such errors occur when students do not understand that (1) histograms are based on intervals, and that (2) intervals are a means of data reduction. In other words, students do not associate individual data points with the intervals represented in the histogram (Friel & Bright, 1996).

Chapin and Johnson (2000) note that students often confuse histograms and bar graphs because they look similar. In a histogram, the bars represent intervals in a distribution of values such as price, or age, or some other fine-grained value. In a bar graph, bars may represent discrete values, for example, different colors, or brands of soap.

Chapin and Johnson (2000) also note that line graphs and bar graphs are both employed to show data which changes over time, and for this reason students sometimes confuse these as well. As they point out, the line graphs show data that varies continuously, such as height, weight, and temperature. Bar graphs are used to show discrete data such as the number of students, the number of cars, or the number of houses.

Strategies for Addressing Student Misconceptions About Graphs

- **Grounded competence.** Kalchman and Koedinger note: “Sometimes mathematics instruction can lead to what we refer to as ‘ungrounded competence.’ A student with ungrounded competence will display elements of sophisticated procedural or quantitative skills in some contexts, but in other contexts will make errors

indicating a lack of conceptual or qualitative understanding underpinning these skills” (2005, p. 389). To address the problem of ungrounded competence they offer three teaching strategies aimed at developing deep understanding of graphing and functions. Kalchman and Koedinger specifically respond to the National Research Council’s 1999 report on how people learn.

The National Research Council (1999) reported on three well-established principles of learning:

1. Students come to the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information.
2. To develop competence in an area, students must have a deep foundation of factual knowledge, a conceptual framework in which to fit those facts, and must organize their knowledge to facilitate its application.
3. Students must be aware of their own learning. In particular, they must analyze new facts and attempt to organize and integrate them with their existing framework.

In response to these principles, Kalchman and Koedinger (2005) offer these strategies:

1. [Start] with a familiar context: Contexts that are familiar to students, such as [a] walkathon, allow them to draw on prior knowledge to think through a mathematical process or idea using a concrete example.
2. [Start] with simple content: To get at the essence of the idea while avoiding other distracting difficulties, our curriculum starts with mathematical content that is as simple as possible—the function “you get one dollar for every kilometer you walk” ($y=x$).
3. [Focus] on having students express concepts in their own language before learning and using conventional terminology. [...] Students may better understand and explain new ideas when they progress from thinking about those ideas using their own invented or natural language. (2005, pp. 389-90)

With statistics, Konold and Higgins (2003) suggest that students benefit from performing all steps of data analysis from beginning to end: question posing, sampling, observing, recording, and displaying. They further suggest that students make graphs of different types from the same set of data. They note that “there are no fixed criteria for judging one data display as superior to another. Rather, the relative value of a plot depends on its intended purpose” (p. 199). *By exploring different types of graphs, students may develop deeper understandings of the connections between data and various displays of that data.*

- **Free-response questions and showing work.** As noted above, Berg and Smith (1994) argue that student comprehension of graphs is closely tied to the process of constructing and explaining them. To that end, the authors recommend presenting students with data and using free-response instruments (such as an axis without labels or scale) and having them explain their work. This allows students to build direct connections between the data and its symbolic representations. Even when students make mistakes, they are more likely to identify and correct them than they would be using multiple-choice instruments.
- **Multiple graphic representations of a single data set.** Bright and Hoeffner (1993) recommend presenting students with multiple graphs, then asking them which graph best represents a specific relationship or situation. Students are encouraged to explain the reasons behind their choices, thus emphasizing the relationship between the data represented in the graph, its structure (axes, labels) and the shape of the graph itself. This approach helps to reinforce graphs as possible representations of relationships within a given data set, as well as disabusing students of the idea that a graph is a picture of an event or situation. A similar strategy for connecting graphs with data is to have students work back and forth from tables to graphs in interpreting and solving problems.
- **Graphing calculators and software.** Students' problems with scale are partly attributable to the traditional approach of drawing graphs using pencil and graph paper, in which students seldom have the opportunity to manipulate graphs after they are constructed. The increased availability of graphing software and calculators offers this opportunity, and can significantly improve students' understanding of the relationship between the scale of a graph and its shape (Bright & Hoeffner, 1993; Dunham & Osborne, 1991).
- **Graphing functions.** With respect to graphing functions, Dunham and Osborne (1991) suggest two strategies for helping students better connect the line graph to the data it represents.

Cornering. In order to help students understand points as ordered pairs that are “projected” from the axes, they suggest a technique called “cornering,” where a right-angled piece of paper is placed with the top right-hand corner touching the point of interest. The paper thus intercepts the X and Y axes perpendicular to the point, reflecting the values of the ordered pair.

“Packing” points. To address the misconception that the line graph of a function simply connects a series of points, the authors recommend a strategy of “packing” points. This approach calls for students to calculate a series of ordered pairs within a particular interval (e.g., X is between 1 and 5) using the equation for the function, plotting them as they go. For each set of order pairs, students are asked to calculate and plot a value that falls between them. In this way, students “fill in” the line with increasingly small differentiations among ordered pairs, rather than drawing a line connecting two or more points. This exercise can be repeated as often as necessary

for the line graph of the function to emerge. A complementary strategy calls for students to plot the line by calculating order pairs randomly rather than sequentially. In other words, rather than plugging in X values in sequence, (1,2,3,4...), students calculate ordered pairs using random values of X (8, 3, 11, 6...). This random distribution of ordered pairs allows the line graph of the function to emerge more slowly, and prevents students from simply “connecting the dots.”

Conclusion

Students' success with graphs will depend on their ability to interpret graphs and connect them to data, or to equations, or problem descriptions. Many have noted that students are more likely to understand graphs that they have created themselves. Similarly, students are more likely to achieve understanding and make connections in problem contexts they are familiar with.

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