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PCK Tools

Measurement: Student Misconceptions and Strategies for Teaching

Introduction

Measurement first appears in the curriculum in the early grades when students attempt to compare the lengths of objects that cannot be placed next to each other. Lehrer, Jaslow, and Curtis (2003) hold that measurement is the gateway to arithmetic and that it helps students understand spatial structure. Measurement teaches children about the relationship between numbers and is the foundation for mathematical concepts like fractions. More importantly, measurement is a connector between the abstract world of mathematics and the concrete physical world to which all students can relate.

State Standards

New Jersey Core Curriculum Content Standards hold the following expectations for what middle-school students should know and be able to do regarding the concept of measurement:

6th grade

Units of Measurement

1. Select and use appropriate units to measure angles, area, surface area, and volume.
2. Use a scale to find a distance on a map or a length on a scale drawing.
3. Convert measurement units within a system (e.g., 3 feet = ___ inches).
4. Know approximate equivalents between the standard and metric systems (e.g., one kilometer is approximately 6/10 of a mile).
5. Use measurements and estimates to describe and compare phenomena.

Measuring Geometric Objects

1. Use a protractor to measure angles.
2. Develop and apply strategies and formulas for finding perimeter and area.
    - Triangle, square, rectangle, parallelogram, and trapezoid
    - Circumference and area of a circle

3. Develop and apply strategies and formulas for finding the surface area and volume of rectangular prisms and cylinders.
4. Recognize that shapes with the same perimeter do not necessarily have the same area and vice versa.
5. Develop informal ways of approximating the measures of familiar objects (e.g., use a grid to approximate the area of the bottom of one's foot).

7th grade

Units of Measurement

1. Solve problems requiring calculations that involve different units of measurement within a measurement system (e.g., 4'3" plus 7'10" equals 12'1").
2. Select and use appropriate units and tools to measure quantities to the degree of precision needed in a particular problem-solving situation.
3. Recognize that all measurements of continuous quantities are approximations.

Measuring Geometric Objects

1. Develop and apply strategies for finding perimeter and area.
    - Geometric figures made by combining triangles, rectangles and circles or parts of circles
    - Estimation of area using grids of various sizes
2. Recognize that the volume of a pyramid or cone is one-third of the volume of the prism or cylinder with the same base and height (e.g., use rice to compare volumes of figures with same base and height).

8th grade

Units of Measurement

1. Solve problems requiring calculations that involve different units of measurement within a measurement system (e.g., 4'3" plus 7'10" equals 12'1").
2. Use approximate equivalents between standard and metric systems to estimate measurements (e.g., 5 kilometers is about 3 miles).
3. Recognize that the degree of precision needed in calculations depends on how the results will be used and the instruments used to generate the measurements.
4. Select and use appropriate units and tools to measure quantities to the degree of precision needed in a particular problem-solving situation.
5. Recognize that all measurements of continuous quantities are approximations.
6. Solve problems that involve compound measurement units, such as speed (miles per hour), air pressure (pounds per square inch), and population density (persons per square mile).

Measuring Geometric Objects

1. Develop and apply strategies for finding perimeter and area.
   - Geometric figures made by combining triangles, rectangles, and circles, or parts of circles
   - Estimation of area using grids of various sizes
   - Impact of a dilation on the perimeter and area of a 2-dimensional figure
2. Recognize that the volume of a pyramid or cone is one third of the volume of the prism or cylinder with the same base and height (e.g., use rice to compare volumes of figures with same base and height).
3. Develop and apply strategies and formulas for finding the surface area and volume of a three-dimensional figure.
   - Volume—prism, cone, pyramid
   - Surface area—prism (triangular or rectangular base), pyramid (triangular or rectangular base)
   - Impact of a dilation on the surface area and volume of a three-dimensional figure
4. Use formulas to find the volume and surface area of a sphere.

The Texas Essential Knowledge and Skills (TEKS) standards set these expectations:

6th grade:

Measurement. The student solves application problems involving estimation and measurement of length, area, time, temperature, volume, weight, and angles. The student is expected to

(A) estimate measurements (including circumference) and evaluate reasonableness of results;

(B) select and use appropriate units, tools, or formulas to measure and to solve problems involving length (including perimeter), area, time, temperature, volume, and weight;

(C) measure angles; and

(D) convert measures within the same measurement system (customary and metric) based on relationships between units.

7th grade:

Measurement. The student solves application problems involving estimation and measurement. The student is expected to

(A) estimate measurements and solve application problems involving length (including perimeter and circumference) and area of polygons and other shapes;

(B) connect models for volume of prisms (triangular and rectangular) and cylinders to formulas of prisms (triangular and rectangular) and cylinders; and

(C) estimate measurements and solve application problems involving volume of prisms (rectangular and triangular) and cylinders.

8th grade:

Measurement. The student uses procedures to determine measures of three-dimensional figures. The student is expected to

(A) find lateral and total surface area of prisms, pyramids, and cylinders using concrete models and nets (two-dimensional models);

(B) connect models of prisms, cylinders, pyramids, spheres, and cones to formulas for volume of these objects; and

(C) estimate measurements and use formulas to solve application problems involving lateral and total surface area and volume.

Measurement. The student uses indirect measurement to solve problems. The student is expected to:

(A) use the Pythagorean Theorem to solve real-life problems; and

(B) use proportional relationships in similar two-dimensional figures or similar three-dimensional figures to find missing measurements.

Measurement. The student describes how changes in dimensions affect linear, area, and volume measures. The student is expected to:

(A) describe the resulting effects on perimeter and area when dimensions of a shape are changed proportionally; and

(B) describe the resulting effect on volume when dimensions of a solid are changed proportionally.

The NJ Core Curriculum Content Standards (NJDE, 2004) go beyond those set by NCTM to ensure that students have a deeper understanding of measurement. New Jersey wants students to understand:

- that all measurements are approximate and that the precision of a measurement is based not only on the tools used to measure, but also on the accuracy required by the activity;
- that shapes with the same surface area do not have the same volume;

- there is an inverse relationship between the unit of measure and the magnitude of a measure;
- that the relationship of a pyramid to a prism and a cone to a cylinder is one third; and
- the impact of dilation on the measurement of both two and three dimensional objects.

Definitions of Measurement

Various authors have constructed different meanings for measurement. Schwartz (1996) defines *measurement* as a three-step process:

- First choose the attribute of an object that is to be quantified.
- Then choose an appropriate unit for measuring that attribute.
- Finally determine the magnitude, or number of units needed, of the measure given that unit.

Schwartz adds that "central to the measurement act is the need for a judgment to be made about the adequacy of the precision for the context at hand" (p. 9). A large part of measurement is choosing the correct unit for the object and task at hand, and being able to measure that object with the precision called for by the task.

Grant and Kline (1993) describe measurement in a similar manner, but they point to some underlying notions that students must understand to become proficient with measurement. They suggest that children need to be able to

- determine an appropriate unit of *measure* for a task;
- decide how to use that *measure* and iterate without spaces (tiling)—both by physically arranging the units of iteration and conceptually understanding how they fit together;
- figure out what happens if the units do not work out evenly;
- understand the inverse relationship between the size of the unit and the magnitude of the measure (the bigger the unit, like feet to inches, fewer iterations will be needed).

Some Important Concepts

Table 1 (Lehrer, Jaslow, & Curtis, 1993) provides useful definitions that clarify the language used to discuss measurement. Unit and iteration are the foundation of measurement—they make up steps two and three of Schwartz' basic definition. The *unit* is a benchmark used for comparison. Units can be either *standardized*, like inches or acres, or *nonstandard*, such as the length of a person's foot or the distance from one's home to school. Since a unit rarely matches the size of the attribute being measured, many units may be needed to be put side by side to accurately describe the size of the object being measured. This is the act of *iteration*. It is critical that each unit in the iteration be identical. A ruler is a good example of the iteration of an inch. On a ruler, an inch is iterated 11 times so that there are 12 identical inches side by side. Another important idea that is revealed by a ruler is that a unit can be *partitioned*. Not only is an inch a partition of a foot, but a quarter inch, shown by hash marks between inches on a ruler, is a partition of an inch. All units can be further subdivided. Lastly, when measuring one counts the number of units iterated as the measurement itself. This is the property of *additivity*.

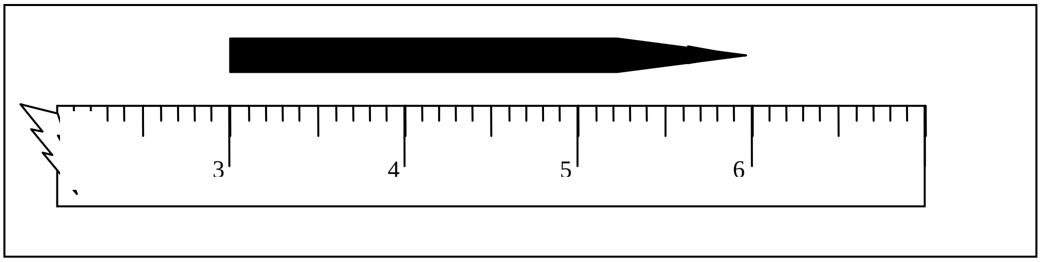
Table 1. Central Concepts in Linear Measurement

	Idea	Description
Conceptions of Unit	Iteration	A subdivision of a length, identified as a unit, is translated to obtain a measure.
	Identical Unit	Each subdivision is identical.
	Tiling	Units fill the length.
	Partition	Units can be divided into smaller units.
	Additivity	Measures are additive, so that a measure of 10 units can be thought of as a composition of 8 units and 2units.
Conceptions of Scale	Zero-point	Any point can serve as the origin or zero point on the measurement scale.
	Precision	The choice of units in relation to the object determines the relative precision of the measure. All measurement is inherently approximate.

These ideas are also important in student understanding of area and volume, especially for understanding that the measurement of those attributes is something more than a magnitude or number. When students are taught that the area of a rectangle is length *times* width, they do not necessarily recognize that one can also find that area by arranging an array of unit squares that sufficiently cover that rectangle. Reynolds and Wheatley (1996) discuss what students must understand before they can appropriately tile to find length, area, or volume. For area, the *tiling* must make a two-dimensional array. An *array* is a physical reorganization of a region into equal rows and columns. To accomplish the task of tiling a region, students must first recognize that big areas can be *partitioned* into smaller ones. They must understand that they need to use a *standard unit*. They need to realize that numbers are associated with the unit regions, and that the sum of the unit regions will be the total area. Lastly, students must organize the tiling so that the entire area is covered leaving no uncovered spaces. Consistency of unit usage is just as important as choosing the correct unit.

The zero-point is the beginning of the scale of a measuring devise, the starting point used for a measurement. Although students are taught to use a ruler by measuring from the beginning of the hash marks at 0, that is not always possible. The difficulty comes for students when the zero-point does not match up with the 0 on whatever scale is being used to measure (See Figure 1). In addition, when measuring length, area, volume, surface area, time, and angles, it is important that students understand that they are measuring

Figure 1



continuous quantities. Continuous quantities can only be approximated by measuring devises like rulers. Almost nothing is actually 1 foot long; it is more likely to be off by some small amount, like 0.00001232173 feet. To the human eye, anything close to 1 foot is 1 foot, because measurement is inherently approximate.

Another important aspect of students' understanding is *conservation* of measurement, which is the idea that a measurement remains constant regardless of what it is measuring. For example, 6 inches of shoe lace is the same length as 6 inches of phone cord, or measuring a shoe lace with inches or centimeters does not change that object's length. Petitto (1990) says that "the acquisition of conservation of length requires the child to move from an initial conflation of the concepts of position and length to a more differentiated conceptual structure that distinguishes clearly between position and length" (p. 57). Petitto found that children first look only at one end-point value when measuring, then later take into account both the starting (zero-point) and ending points. This form of proportional reasoning is often critically missing with some children at the middle-school level, and even at the high-school level.

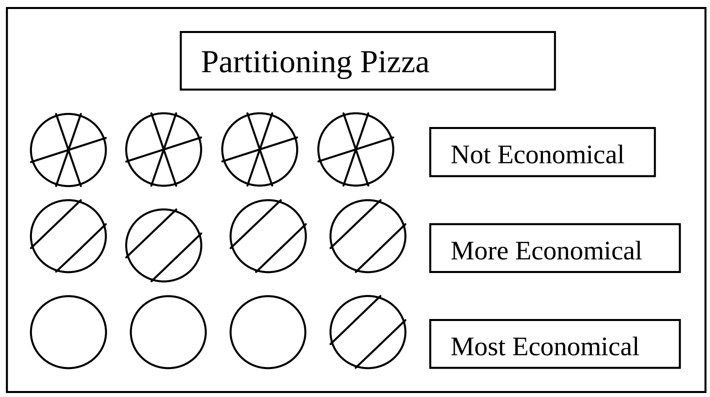
Development of the Concept of Measurement

We must to look at the early grades to understand the source of the most common student misconceptions of measurement in the middle grades. Much of the research on measurement has been conducted at the primary level, but the findings seem applicable to all levels. Data from the Third International Mathematics and Science Study, or TIMSS (see Beaton, et al., 1996), show that many students at the middle-school level have difficulty with basic measurement properties of length and area, and that students in the United States lag behind the low international averages in the domain of measurement. These same deficiencies can be seen in data from the National Assessment of Education Progress, or NAEP[1] (see Hiebert, 1981; Kouba, et al., 1988; Strutchen, et al., 2003). In the following discussion, we will look at the development of measurement ideas and research on student misconceptions of measurement of length, area, surface area, volume, angle, and others.

[1] Unless otherwise noted, all data from the NAEP and TIMSS is from the eighth grade level

Lamon (1996) studied the development of unitizing. She holds that while the "decomposition of a given whole into small units appears to happen immediately and naturally ... reunitization into composite pieces ... develops less rapidly" (p. 188). In other words, the development of measurement begins by partitioning a length into unit measures. Children first learn that they can subdivide a whole into smaller pieces, and then they learn that they can combine those smaller pieces into more useful chunks. A composite unit is any collection of smaller units that simplifies the act of partitioning, making it more economical. Using the cent as a unit to partition, a child can break $.75 into 75 cents. A child with more sophisticated measurement abilities can also use 5 cents as a unit, as well as 10 cents, or even 25 cents. Although 15 cents could also be used as a composite unit, Lamon suggests that "student partitioning strategies were situationally specific and showed a strong observance of social practice and practicality" (p. 188). There is no 15-cent coin, so students would not be likely to use that composite unit. Lamon also describes these more sophisticated partitioning strategies as becoming more economic. Not only do students start using larger units, when dividing objects like pizza, they will make fewer and fewer cuts to evenly distribute the pizza. An example is that if 4 pizzas are to be split between 3 people, children will first cut the pizzas into 6 slices and distribute evenly, then 3 pieces, and lastly, only 1 of the pizzas will be split three ways and each person will be given a whole pizza and one additional slice. (See Figure 2) The more sophisticated the strategy, the fewer the cuts and therefore the more economical it is.

Figure 2



Another important development in measurement suggested by Lamon (1996) is that children make a conceptual change in understanding when they realize that measurement asks the question "How much?" not "How many?" Schwartz (1996) looks specifically at this issue, examining how measures of discrete quantities can be answered by asking the question, "How many?" but continuous quantities require the more specific question, "How much?" Schwartz describes the first act as counting and the second as measuring. He is quick to point out that most quantities can be described as being both discrete and continuous. Fifty pounds of nails is a continuous quantity, but that same collection of nails could be described discretely as 100,000 nails. Regardless, young children think of measurement only in terms of discrete value and only later qualify that value.

Development of Concepts Related to Length Measurement

To understand length, Lehrer (2003) suggests that students must first learn that a count of so many units corresponds to a length of so many units. Students may not understand that by counting 3 inches they have a measurement of 3 inches. Students do not automatically perceive that the number they obtain represents a length or a volume. In addition, to find a measurement a student must first understand the zero-point and the unit. Included within the meaning of the unit are the procedure for iteration, the need for identical units, the inverse relationship between unit size and magnitude, and that a unit can itself be subdivided. Lehrer posits that most youth become proficient with these concepts at the primary level.

Barrett, et al., (2003) point out that children's length measurement strategies do not always match the procedures of unitization and iteration. They describe the development of length measurement in three stages:

- Students first compare objects visually without using units.
- Students then compare objects using first nonstandardized units and then standardized ones.
- Students finally employ multiplicative strategies of composite units.

It is in the middle stage when students stop counting the hashes on a ruler (position) and start counting the spaces (length). The misconception of counting the hashes instead of the spaces will be discussed later. Shadowing the findings of Lamon (1996), Barrett, et al., (2003) found that there was a progression from using more exact units and then using multiplication of composite units to simplify the process.

Development of Concepts Related to Area and Volume Measurement

One finds that children initially think that the measurement of area and length are the same. Because of this, development of strategies for area measurement often lags behind those for length measurement. Lehrer (2003) points out that the conventional strategies for determining the measurement of length may impede a student's ability to restructure a region into an array of rows and columns to determine its area.

Battista (2003) suggests that "the foundation for developing competence with measuring area and volume in standard measurement systems is understanding how to enumerate meaningful arrays of squares and cubes" (p. 122). Area is seen as an array of rows and columns ($l \times w$), and volume is seen as an array of row, columns, and planes or layers of rows and columns ($l \times w \times h$). Battista proposes four mental processes that are needed for students to understand and adequately represent area and volume as arrays:

- Children need to be able to form and use mental models.
- Children need spatial structuring abilities to orderly partition a shape.
- Children need to be able to locate a unit with an array—in other words, portioning the shape in an organized manner where each unit shape can be accounted for.
- Children need to be able to organize their units by composites (e.g., rows or layers of rows and columns).

As children get older, one finds that they slowly master these four processes in this order.

Research by Battista, et al., (1998) on youth understanding of three-dimensional arrays found that students first see arrays as unorganized sets of cubes, then they begin to structure the shapes locally, and lastly globally. By *local*, they mean that children first look at the structure of a cube by looking at the faces separately. By *global*, they suggest that the students then organize volume by row and columns and layers of rows and columns. In a subsequent study, Battista, et al., took these findings and tried to explain them through conducting an experiment with two-dimensional arrays. They found that there is the same progression with both two- and three-dimensional arrays. With area, students first counted an array by taking a one-dimensional path around the array to count the squares (often ending in an incorrect count). Then students structured the area locally, separating the shape into the sides and the interior (also with incorrect counts). The most sophisticated method for counting the squares in an array, coming last developmentally, was to organize the area globally, making uniform rows and columns.

General Misconceptions Concerning Measurement

Measurement as an Abstract Concept

Lehrer (2003) holds Euclid responsible for the modern divorce between the fields of geometry and measurement. Although geometry was born out of measurement, Euclid made a point of using a straight edge and a compass to make his famous theorems rather than a ruler and a protractor. The field of geometry is therefore more abstract, and more generalizable. While measurement is inherently imprecise, geometry is quite the opposite. This same divorce between the physical world and the world of mathematics can be seen with children learning about measurement. To them, measurement is an abstraction on the physical world. Lehrer, Jaslow, and Curtis (2003), restating the ideas of Piaget, suggest that “units of measure may be seem transparent to children because one can literally point to them on measurement devices like rulers, but understanding the nature and properties of units is the end point of a long process of learning” (Piaget, Inhelder, & Szemisnka, 1960) (pp. 100-101). Since measurement units have no meaning to young children, measurement is quite abstract to them. The authors continue that even students who are good with rulers do not necessarily understand units and iteration.

Resistance to Displacing Unsuccessful Intuitive Strategies

In a study by Chui (1996), it was found that youth often used intuitive strategies to measure objects rather than the strategies that they had been shown were successful. In a task where students were asked to decide which of a number of linear paths were longer (some straight, some curved, and some crooked), they were given a chance to try their own method, often unsuccessfully, and then were shown an effective strategy. On a second trial, 93% continued to use their unsuccessful strategies rather than the recently proven one. The author is quick to defend student use of intuitive strategies because they can sometimes be useful and experts also use them, but these strategies are often formed before students come to school, they can conflict with expert (teacher) strategies, and children often resist attempts to displace them. Students’ intuitive strategies are frequently incorrect and these strategies are often difficult to suppress.

Conflict Between Adjectival and Nominal Quantities

Schwartz (1996) makes the argument that one of the reasons measurement is so difficult for children is because in math classes children often work with numbers stripped of their significance. He offers an analogy to grammar. Numbers are often treated as nouns in a math class (add two and three), but numbers act as adjectives in the physical world (3 apples). Whereas most people assume that there is a one-to-one correspondence between the two worlds, Schwartz argues that adjectival (adjective) quantities do not follow all of the properties that nominal (noun) quantities do. A brief example is with the property of additive inverses, where $2 + 3 = 5$ can be undone; $5 – 3 = 2$. However, 2 apples plus 3 oranges equals 5 fruit, but if you take away 3 oranges from 5 fruit, you do not necessarily have 2 apples. Schwartz goes on into issues of addition and multiplication that cause difficulties when switching between nominal and adjectival quantities. With multiplication of measures, the numbers and the referents change; this does not occur with addition. For example, 3 multiplied by 3 gives 9, but 3 inches multiplied by 3 inches give 9 square inches. Schwartz argues that there is a kind of common sense that children need to acquire to determine which rules from nominal situations apply to adjectival ones.

Difficulty in Pulling Accurate Measurements

Kouba, et al., (1988) interpreted results from measurement items on the fourth NAEP as showing that "performance on items that could be solved visually was higher than on those that required more abstract thinking" (p. 15). In other words, the more visual aids, such as diagrams, that students were given to solve measurement problems, the more likely they were to answer correctly. Hiebert's analysis (1981) from the first NAEP suggests that students in the middle grades have difficulty switching from whatever unit a problem started with. He also found that students have difficulty reading graphs and pulling accurate measurements from tools such as a thermometer. On a problem involving a pictograph, students failed to read the entire problem and did not learn that each human in the pictograph represented 10 people. On the thermometer problem, the scale read 58 degrees, but the most common answer was 59; only the 10s degrees were labeled on the instrument with hash marks every 2 degrees. Hiebert suggests that when students are unable to come up with the correct answer they often choose the most immediate or salient number(s) in the problem. His overall analysis suggests that kids are proficient with simple measuring, but lack understanding of underlying concepts.

Misconceptions Relating to Length

Clements (1999) found that less than 50% of 7th graders could determine the length of a line when the ruler was not aligned with the object starting at zero. Similar results are reported by both the NAEP and TIMSS. Although length measurement is something assumed by the national and state standards at the middle-school level, it is something that is difficult for many students. The issue with the broken ruler problem is that students will look at the end number on a ruler that lines up with the object without referring to the zero-point, or where the object starts on the ruler. (See Figure 1 above) Clements explains his results by suggesting that "most students use measurement instruments or count units in a rote fashion and apply formulas to attain answers without meaning" (p. 5). Since children learn to use a ruler with the beginning of the object starting at the 0 on the scale, they continue to believe that all measurements with a ruler work the same way. They learn to read the end point on the ruler as the measurement. Like Clements, Outhred, et al., (2003) suggest that children follow procedural methods when finding lengths. They suggest that children count units, count marks, or count the spaces; it is when children count the marks that they make mistakes. When students count the hashes, they always get the measurement off by one unit because there is always an additional hash mark. Here, the student who makes this mistake may not understand that length is the space between the hashes.

Outhred, et al., (2003) also found that children could not generalize measurement methods to more practical problems. They believe that reliance on textbooks is responsible for students not developing more abstract concepts like scale. Moreover, the authors point out that when students mentally combine counting and measuring, they encounter problems when they attempt to measure continuous quantities. A reliance on counting impedes a student's ability to see that units can be subdivided. The authors argue that measurement is difficult for students because the youth conflate counting and measuring. Furthermore, understanding of length measurement is a prerequisite for area, volume, and spatial measurement (Izsak, 2005; Outhred, et al., 2003).

Petitto (1990) quotes Fuson (1984) to suggest that young children rarely understand that "numbers are represented on the numberline by lengths—instead, numbers are thought to be represented by the points they label" (p. 219). Petitto continues, proposing that children see numbers on a ruler as an ordered sequence, not as the spaces between those numbers. Again, there is a disconnect between the number that children find when measuring, and the meaning of that number that many teachers and students take for granted. Kilpatrick, et al., (2001) copy these sentiments and add that teachers should not assume that their students understand the number line. The number line contains many implicit properties that mimic those of measuring instruments, such as identical units, iteration, tiling, partition, and zero-point. One huge difference, however, is that while a ruler is inherently imprecise, the number line is the definition of precision.

Strutchens, Martin, and Kenney (2003) analyzed the 2000 NAEP data and found that measurement knowledge is superficial; students could choose the correct tool or unit, but they could not use them well. They found that 67% of 8^{th} graders were able to get the misaligned ruler problem. They also found that while 67% of the students could accurately find the length of one side of shape given its perimeter, only 6% could accurately explain why one shape out of three had the longest perimeter. TIMSS numbers are a little more disconcerting. Results show that only 52% of 8^{th} grade students could correctly work through a more complex mismatched ruler problem. Additionally, 23% of the students could find the perimeter of a square given its area, and 22% could identify the ratio of the side of a shape to its perimeter.

Misconceptions Relating to Area and Surface Area

Lehrer (2003) points out that since area is often treated as a multiplication of lengths, children do not recognize the product as a measurement. Instead, children see area as a calculation, a number separate from a real world situation from which the original measurements—length and width—were taken. Outhred, et al., (2003) contend that a unit of area is more conceptually difficult to understand for children than a unit of length. To measure area, students are often taught to tile over a region with a unit square. However, Battista, et al., (1998) point out that the row and column structure of an array is not naturally understood by children and needs to be personally constructed by each individual. Lehrer, Jaslow, and Curtis (2003) suggest that the foremost challenge of area measure is the restructuring of the plane into an array of units.

Two major problems occur when students attempt to tile over a region. First, children often try to tile a region with units of a similar shape to that region (see Lehrer, 2003; Kilpatrick, et al., 2001; Lehrer, et al., 2003). For example, when asked to measure a hand, children will insist on using beans rather than squares, even though beans do not meet the space filling property required for tiling. Additionally, students will not violate any boundary of the region when they are measuring. If a student were to use a square unit to estimate the area of a hand, they would only use squares that fit entirely within the border of the hand. These are two intuitive strategies that students use when tiling that are hard to overcome.

Kordaki and Potari (1998) found that children often believe that the ratio between corresponding areas is the same as the ratios between corresponding dimensions of those regions. Using this logic, if children see two squares, one with double the area of the other, many assume that all of the sides of the larger square would be double of those in the smaller square. This relates directly to the New Jersey standards, insisting that students should understand the results of dilation on area and volume.

Research on surface area has produced similar findings. Bonotto (2003) found that although children know a lot about surfaces outside of school, in school surface area is only realized on the numerical level. She believes that traditional teaching separates the classroom from real world experience. She feels that in teaching area, teachers use a formula to replace measurement with an instrument and this impedes student understanding of the meaning and role of measurement units. Research suggests that a poor understanding of area is related to rote practice with formulas as well as students not being given enough time to develop an understanding of measurement. Again, procedural and rote memory tasks get in the way of students having meaningful experiences with measurement.

NAEP & TIMSS. Kouba, et al.'s findings (1988) from the 4th grade NAEP suggest that one-third of students confused area and perimeter and that performance on area and perimeter on the test was below 50%. Fewer than 40% of students could estimate the area of an irregular shape superimposed on a grid, and only 10% could find the area of a shape if told the shape was a square and the length of one side of that square. Strutchens, Martin, and Kenney's analysis (2003) of the 2000 NAEP found 78% of students could tell the area of a more regular shape on a grid and 67% could draw a shape of 12 units squared.

TIMMS data suggests that students had difficulty finding the area of irregular shapes composed of regular ones. In addition, only 40% of students at the 8th grade level could accurately measure the area of a parallelogram given all dimensions and a diagram that suggested a possible algorithm.

Misconceptions Relating to Volume and Angle

Less research has been conducted on the remaining areas of measurement. As for volume, Outhred, et al., (2003) suggest that the measurement of volume is very abstract. In fact, most tools for measuring volume, such as a measuring cup, have a linear scale to read off, making it more difficult for students to accurately conceptualize what attribute they are measuring. Volume problems are often seen as filling problems and not as ones where students have to pack a space with unit cubes. When students are shown that they can measure volume by packing with unit cubes, many only count the cubes that are visible from a given angle (Outhred, et al., 2003; Lehrer, 2003).

As for angles, Lehrer (2003) suggests that there are key two issues with this measurement. The first issue is that students believe that the length of the rays constructing the angle has an effect on the measurement of the angle. Children think that the longer the rays, the larger the angle. This may be due in part to how children measure angles, almost always measuring from the end of the rays. On this same point, children often believe that angle measurement is another type of length measurement and rarely understand that an angle is a measurement of rotation. The second issue is that children believe that the angle's position in space has some effect on the measurement of that angle.

NAEP and TIMSS. Few items on the 2000 NAEP dealt with surface area and volume—only three—and the rates of correct answers were below 40%. As for angles, Strutchens, Martin, and Kenney (2003) report that 8th grade students had problems recognizing an angle smaller than a right angle but 67% could accurately order a set of five angles by size. TIMSS data suggests that students internationally had difficulty picking out an angle that was 45 degrees out of a set of four angles.

A Summary of the Research

General	• Measurement and measurement units are abstract concepts for children. • Students make use of intuitive strategies that are frequently incorrect and are difficult to suppress. • Students find the concept of measurement difficult because they are unable to understand and distinguish between the nominal and adjectival natures of numbers.
Length Measurement	• Students have difficulty in determining the length of a line when the ruler is not aligned with the object starting at zero. This is because they count the marks/hashes on the ruler rather than the units. • Measurement is difficult for students because they conflate counting and measuring. • Children rarely understand that numbers are represented on the number line by lengths; instead, numbers are thought to be represented by the points they label.
Area and Surface Area	• Children see area as calculation—a number separate from the real-world situation—rather than a measurement. • Children have difficulty in tiling a region with a unit of different shape than the region and they will not violate any boundary of the region while tiling. • Children often think that the ratio between two areas is same as the ratio between the dimensions of the regions. • Formula replaces the measurement with an instrument and impedes the understanding of measurement units.
Volume and Angle	• When students have to find the volume of a space filled with unit cubes, they usually count only the cubes that are visible from a given angle. • Students think that the length of the rays constructing the angle has an effect on the measure of the angle. • They rarely understand that angle is a measure of rotation.

Strategies for Addressing Student Misconceptions

General

Lamon (1996) says students need more experience with unitizing and partitioning, especially in the middle grades. She proposes that students should work with these strategies until they start to use more economic strategies:

> Students need extensive presymbolic experiences involving these conceptual and graphical mechanisms [unitizing and partitioning] in order to develop a flexible concept of unit and firm foundation for quantification, to develop the language and imagery needed for multiplicative reasoning, and to conceptually coordinate the additive and multiplicative aspects of rational numbers. (p. 192)

In other words, children need a good amount of experience working with units, iteration, and partition. To help students better understand units, giving them a nonstandard strip of paper and letting them measure objects through a mechanical iteration can help to show students why it is the space, and not the hash, that is doing the measuring on a ruler. Further, they might be asked to design their own ruler using a nonstandard unit, thus focusing attention on the role of units in the measuring instrument (Van de Walle, 2001). To help students understand partition and units, a teacher can give long strips of paper to a class and ask them to measure objects shorter than that benchmark that was handed out. It is important to allow the children to explore their own solutions and be able to explain those solutions to other students. This practice gives the students more ownership of their knowledge.

Teachers can aid in the process by asking students questions that force them to use more economic strategies, such as making the numbers in the problem so large that the students might have to change their strategy. Lehrer (2003) suggests that for children to be able to partition a unit, they need opportunities to split up lengths and areas on their own.

Hiebert (1981) and Strutchens, Martin, and Kenney (2003) suggest that students need experience choosing their own units of measurement and working with nonstandard units to be able to generalize measurement strategies and properties such as scale. Hiebert also believes that students need to estimate. Adams and Harrell (2003) suggest that to improve students' estimation strategies,

- youth should be given time to build the skill;
- youth should learn to use multiple senses when estimating;
- youth should rely on prior knowledge (meaningful reference points); and
- youth should take multiple steps when estimating, gathering all information before making an attempt.

Support for estimation is seconded by Joram (2003), who suggests that children's ability to use benchmarks to estimate measurement will improve their overall measurement sense;

something she defines as “having a ‘feel’ for units of measurement and possessing a set of meaningful reference points or benchmarks for these units” (p. 57). She holds that students in the middle grades have little knowledge of measurement of everyday objects and that because of this their ability to estimate is inconsistent. Joram argues that benchmarks, or meaningful reference points, can be used both for estimation and for students to learn about measurement systems and principles. She also warns that benchmarks of similar units need to be connected so that children understand that a thumb is $1/12^{th}$ of a human foot, just as standardized units need to be connected.

Teaching Length

While many authors suggest that students should first work with nonstandard units to measure length, Clements (1999) suggests the opposite. He believes that children should use standard measurement tools early, even though they may not completely understand them. He has found that younger children prefer them, and they give the youth the advantage of an early start for internalizing benchmarks based on standardized units. While it is important to have nonstandard units for learning about the properties of measurement, standard units are the basis for estimation and relating school to the physical world.

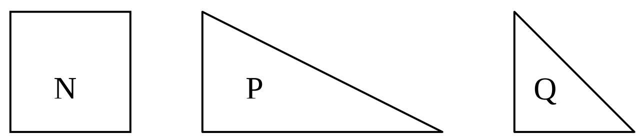
To improve student use of rulers and number lines, Barrett, et al., (2003) offer a number of suggestions to help students look at the spaces rather than the hashes. They advise that when using nonstandard or standard units, the student should decide on the unit, label the zero-point, and explain how to iterate the unit. Most importantly, teachers need to ask children questions that guide them to use more sophisticated strategies that the students devise themselves. For example, if a student counts hashes instead of spaces, give that child a string the length of one unit, and ask for the measurement of the string. When the child gives a measurement of two units, ask the child what a measurement of one unit would look like.

A way to make measurement more meaningful for students is to have them work with familiar objects. In addition to being more meaningful to the students, it will be more practical because this will give them a repertoire of familiar objects that they can use as benchmarks to estimate measurements in the future. Another suggestion is that when a teacher finds that many students in a class are coming up with different answers, it is often beneficial to have the students explain their answers to each other and have the entire class discover the correct solution through mathematical reasoning. The teacher can act more like a moderator of a group conversation, organizing the discussion and directing it towards a more valuable outcome. It is also critical that students understand the importance of naming their units. Students not only need to become accustomed to using adjectival quantities (3 *inches* instead of 3), but they need to be asked to explain why they chose to use certain units. School mathematics is often numbers devoid of meaning, and measurement is one of the places where teachers and students must make connections between school and the physical world.

Understanding Perimeter

As perimeter is defined as the total length around a region, we can expect students to have difficulties in understanding concepts related to it, similar to those they experience with measuring length. Martin and Strutchens (2000) analyzed a small set of responses to a 1996 NAEP question asking students to compare the perimeters of three paper shapes given to them. (See Figure 3) In their analysis, they found that some students simply wrote that *P* looked like it had longer sides.

Figure 3



(These shapes were available for use as manipulatives.)

Which of the shapes N, P and Q has the longest perimeter?

Shape with longest perimeter: ____________

Use words or pictures (or both) to explain why.

They suggest that teachers should encourage correct response strategies such as

1) drawing a line segment for each shape by tracing the length of each side of the figure. This tactic of "unrolling" the sides of a figure allows the use of a comparison strategy, which helps the student grasp the basic concept of perimeter.

2) one used by an 8th grade student to solve the problem in Figure 3. The student created a measurement system apparently based on assigning a length of 3 units to a side of square *N*. The student then assigned values to sides of shapes *P* and *Q* on the basis of how they compared to the side of *N*. Then he/she calculated the perimeters of *N*, *P*, and *Q* and correctly concluded that *P* has the largest perimeter. This flexible use of units suggests a good understanding of units in relation to the measurement of perimeter.

Students should be able to build on informal experiences like *counting how many footsteps it will take to walk around the play ground* or *use paper clips laid end to end to find the perimeter of various shapes* to develop more formal calculations of perimeter using a ruler.

In short, many hands-on activities are required to set the concepts firmly in place.

Teaching Area and Angle

Understanding Area and Surface Area. Strutchens, Martin, and Kenney (2003) suggest that instead of memorizing formulae, students need to learn *why* those formulae work and need experiences applying them in a variety of nonstandard contexts as opposed to rote practice on rectangles and triangles.

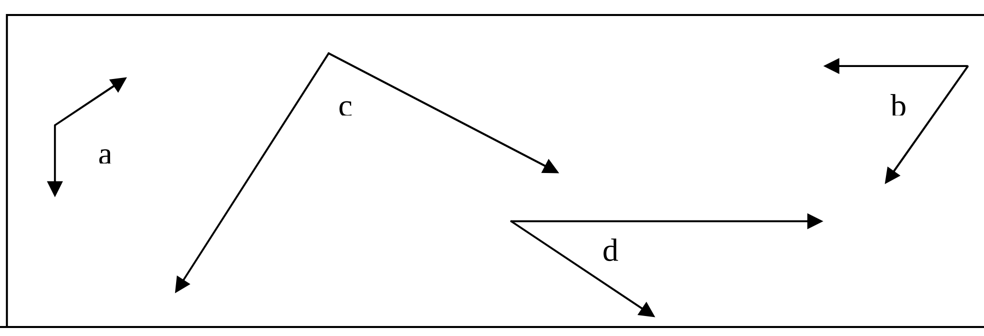
Bonotto (2003) refers to a study by Nunes, Light, and Mason (1993) that found taking direct measurements to have a significant positive effect on students' understanding of surface area. Bonotto seconds this idea when suggesting that surface area measurement become much more real with unit covering followed by formulaic calculation. She feels that the act of unit covering gives "greater value to the knowledge and strategies children possess in practice" (p. 167) and connects school to the physical world.

Students need to be given more time to explore the procedure of unit covering/tiling in order to have "physical experiences" learning about area as a measurement instead of a calculation. Students should also be given experience determining the area of a region superimposed on a square grid. Here, students are forced to think about two common misconceptions that came with unit covering in the research literature—both that they want to use a shape similar to that being measured and that they only count squares that are completely within the region being measured. Again, students need to be asked to explain their methods of measurement to each other. To connect formulas to measurement, it might help to have students both physically measure areas and surface areas, and then compute those same areas using formulas. Through the comparison, students not only can get a better understanding of how formulas work, but they also can see first-hand that measurement is always an approximation. Rarely will students find that the two numbers match exactly.

Understanding Angles. Strutchens, Martin, and Kenney (2003) suggest some approaches that may be useful for teachers to consider. First, students need to have more experiences in qualitatively comparing the size of angles, instead of immediately focusing on the use of a measuring instrument. For example, look at the task in Figure 4:

Figure 4

Order the following set of angles from smallest to greatest.



To successfully complete this task, students need to be able to visualize how much an angle is “bent” or how “pointy” it is. They could trace an angle and superimpose it on top of others to better see the relationship. Furthermore, it is crucial to understand that the lengths of the ray used to represent the arms of the angle do not affect its angle measure. Wilson and Adams (1992) suggest that students may benefit from experiences with nonstandard units of angle measure, such as iterating wedges to measure angles, before they begin using protractors. (See Figure 5) This way they can see how the unit is iterated to fill the angle and that the units are rotated so that the successive units are placed edge to edge.

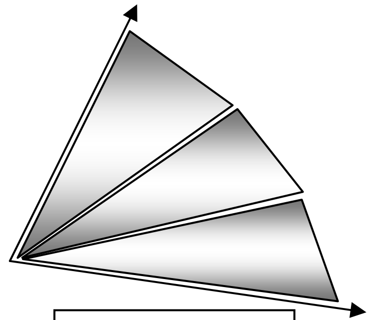


Figure 5

Conclusion

Measurement is a diverse topic that appears in the mathematics curriculum from kindergarten to 12th grade. Research findings point to a number of misconceptions that students carry over from the elementary grades. Even though length measurement is the gateway to arithmetic and the measurement of perimeter, area, surface area, and volume, data from the NAEP and TIMSS show that students at the middle-school level still have many difficulties with measuring length and volume. The number of studies decreases quickly as you go from length measurement to area, to volume, and on. Measurement is important and, as many authors have noted, it is the connection between the familiar physical world and the abstract world on mathematics. Building a strong foundation in measurement should help students to not only better understand measurement in mathematics, but also the world around them.

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