



PCK Tools

Proportional Reasoning: Student Misconceptions and Strategies for Teaching

The ultimate importance of proportional reasoning results from its power to facilitate problem-solving capabilities. One of the most important reasons to make problem solving a central part of the school mathematics curriculum is the contributions that such experiences make to children's understandings of other central concepts, many of which in turn are related to and involve proportional reasoning.

(Lesh, Post, & Behr, 1988, p. 107)

A *proportion* is a statement of equality between two ratios, often written as $\frac{a}{b} = \frac{c}{d}$. Since ratios themselves are a relationship between two numbers, a to b and c to d, proportions can be thought of as a relationship of relationships. Ratios can also be written as a:b, as in the scale of a map, 1":10,000". A ratio can be conceived as many things, such as a rational number, a rate, a quotient, and a fraction. Lesh and colleagues (1988) define *proportional reasoning* as involving "the mental assimilation and synthesis of the various complements of [the ratios in a proportion] and an ability to infer the equality or inequality of pairs or series of such expressions based on this analysis and synthesis. It also involves the ability to generate successfully missing components regardless of the numerical aspects of the problem situation" (p. 93). This quote is quite dense, but it sums up the following discussion. We will spend the rest of the paper breaking down the typical problems described within this quote—comparison and missing value—and the thinking exhibited by middle-school students while working through such problems.

A study by Reaner (1977) found that college freshmen, while skilled at arithmetic computations, showed little knowledge of proportional reasoning. Tourniaire and Pulos (1985), in a literature review of studies on proportional reasoning, hold that adults are equally poor at proportional reasoning. Lamon (2007) believes that 90% of adults do *not* reason proportionally. Proportional reasoning is a complex topic in mathematics, the study of which is generally drawn out over the middle grades, and never fully understood. NAEP scores are consistently low with regards to proportional reasoning problems; these results also hold for American students in the international TIMSS math assessments. In the research, consistent use of proportional reasoning by a majority of students has not been found outside of the upper grades of high school in wealthy, suburban communities with students in upper track mathematics classes.

Yet, as the opening quote indicates, proportional reasoning has important implications for higher level mathematics, and can be seen as a segue from elementary-school arithmetic to more abstract high-school mathematics. In fact, Lesh, et al., (1988) believe that proportional reasoning should be considered the capstone of elementary mathematics and the cornerstone of high school mathematics. As a capstone, it can be thought of as an application of the number facts and operations learned in grades K through 5, especially addition, subtraction, multiplication, division, equality, and rational numbers/fractions. The authors hold that it is a cornerstone for two reasons. First, there is a process to solving a proportional problem. Instead of simply carrying out an operation like addition or multiplication, students go through a three-step process that is useful later on in algebra, geometry, and calculus: First describe the problem, then transform it, and finally complete a procedure to get an answer. Second, proportional reasoning involves the mathematical concepts of equivalence, variables, and transformations. For example, in elementary school math, an equals sign usually denotes a command, “do this addition,” as in $4 + 35 = ?$. With proportions, the equals sign goes both ways, connoting a relationship of equality.

As we will see, there is also a disconnect between the methods employed by students and those suggested by math textbooks. Textbooks tend to treat proportional reasoning as an opportunity to learn how to set up and solve equations, as proportions are typically dealt with in a symbolic, and often abstract, way. Hart (1984), as part of the Study of Errors in Secondary Mathematics, found that students’ methods when working with ratios were often intuitive, context-bound, not relying on a formal symbolic method, and based largely on counting, adding, and combining, almost always with whole numbers. She also found that these intuitive methods are not easily abandoned, especially because they are more familiar and students can find success using them on easier problems. However, with tasks that are more complex, these solution strategies often fail. As we will see, these findings are replicated by many of the studies discussed below.

In almost no study do researchers find that any of the students use the espoused school method of cross-multiplication to solve proportional problems; yet this is the method that is taught and re-taught in schools. State, national, and international assessments tell us this is a difficult topic for students. Throughout the following discussion, we will not only look at student errors and misconceptions, but also their correct strategies, theories of development, and suggestions from the literature for improving student knowledge and use of proportional reasoning.

State Standards

On proportional reasoning, **New Jersey Core Curriculum Content Standards** hold the following expectations of what middle school students should learn about and be able to do:

- Explore the use of ratios and proportions in a variety of situations.

Understand and use ratios, proportions, and percents (including percents greater than 100 and less than 1) in a variety of situations.

- Understand and use ratios, proportions, and percents (including percents greater than 100 and less than 1) in a variety of situations.
- Solve problems involving proportions and percents.

The **Texas Essential Knowledge and Skills (TEKS)** standards set these expectations:

- Use multiplication and division of whole numbers to solve problems including situations involving equivalent ratios and rate.
- Use ratios to describe proportional situations
- Represent ratios and percents with concrete models, fractions, and decimals
- Use ratios to make predictions in proportional situations
- Use division to find unit rates and ratios in proportional relationships such as speed, density, price, recipes, and student-teacher ratio
- Estimate and find solutions to application problems involving proportional relationships such as similarity, scaling, unit costs, and related measurement units
- Select and use appropriate forms of rational numbers to solve real-life problems including those involving proportional relationships.
- Use multiplication by a constant factor (unit rate) to represent proportional relationships
- Compare and contrast proportional and non-proportional linear relationships
- Estimate and find solutions to application problems involving percents and other proportional relationships such as similarity and rates.

Proportions and Proportional Reasoning

This section is quite lengthy as there are many words and ideas that need to be defined before looking at the research on student use of proportional reasoning. There appears to be a debate within the literature as to the meaning of some of the words defined below. We will use one set of definitions throughout to limit any ambiguity concerning these ideas.

Proportional Reasoning. Lamon (2007), in her handbook chapter on rational numbers and proportions, expands on the earlier definition of proportional reasoning to include “supplying reason in support of claims made about the structural relationships among four quantities in a context simultaneously involving covariance of quantities and invariance of

ratios or products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities” (p. 638). Lamon is bringing specific attention to the relationship of the two numbers on either side of the equals sign, and the unvarying relationship represented by the equals sign. She also points out that proportional reasoning involves detecting, expressing, and analyzing proportional relationships. Proportional reasoning is often conceived as a type of multiplicative reasoning. *Multiplicative reasoning*, in part, is the ability to discern situations that involve multiplicative relationships, and the ability to analyze and apply that information. Lamon believes that proportional reasoners should be able to tell the difference between additive and multiplicative situations in order to apply the appropriate transformation. She describes adding, joining, subtracting, separating, and removing as additive situations and shrinking, enlarging, scaling, duplicating, and fair sharing as multiplicative situations. Remember that division is considered multiplication, just as subtraction is considered addition ($\frac{3}{4}$ can be thought of as $3 \times \frac{1}{4}$ and $3 - 4$ can be thought of as $3 + (-4)$).

Karplus, Pulos, and Stage (1983b) put forward three steps of proportional reasoning (p. 219):

1. Identification of two extensive variables that are applicable;
2. Recognition of the rate of intensive variable whose constancy determines the linear function;
3. Application of the given data and relationships to find (i) an additional value for one extensive variable or (ii) comparison of two values of the intensive value computed from the data.

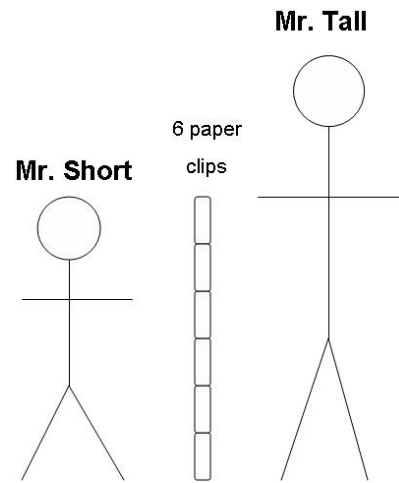
Schwartz (1988) defines *an intensive variable* as one that is ordinarily not counted or measured directly, whereas an *extensive variable* is. Extensive variables describe the quantity that is associated with an entity, like *5 apples* or *20 dollars*. Intensive variables are more qualitative, describing a relationship between two quantities, often using the word “per,” as in *10 miles per hour* or *15 cents per stick of gum*. An intensive variable is a comparison of two extensive variables. Therefore, the first step to proportional reasoning is finding two quantities that you want to compare. The second step is to find the intensive relationship, or ratio, of those two quantities. The last step is to solve the type of problem, either comparison or missing value.

Types of Problems. Proportional reasoning problems always include the creation and analysis of a proportional relationship, $\frac{a}{b} = \frac{c}{d}$. These problems fall into two categories. In the first, *missing value* type of problem, one ratio is set up and one of the numbers in the second proportion is given.

The task is to find the missing value. For example, Karplus and Peterson (1970) created a problem, replicated in many studies, where a pair of stick figures, Mr. Short and Mr. Tall, is measured with two types of implements (see Figure 1). In the original task, the two were measured with large paper clips, Mr. Tall being six large clips and Mr. Short being four.

Students were then given small paper clips and a picture of Mr. Short, who measured six small clips. The students were then asked to predict the height of Mr. Tall in small clips.

Figure 1



The problem is set up as $4:6 :: 6:x$, or as the students are taught to write in school, $\frac{4}{6} = \frac{6}{x}$. This can also be written as $\frac{6}{4} = \frac{x}{6}$.

In the second, *comparison* type of problem, two sets of extensive quantities are given and students are asked to decide which ratio is bigger, or whether they are the same. This type of problem has been shown to be more difficult for students than the missing value type. Noetling (1980a, 1980b), in another much replicated task, had students decide which combination of water and orange juice would taste sweeter (see Figure 2). A question would be posed as such, *Would three cups of water and two cups of orange juice be sweeter or the same as four cups of water and three cups of orange juice?* The students then have to decide whether $\frac{2}{3}$ is larger, smaller, or the same as $\frac{3}{4}$.

Figure 2

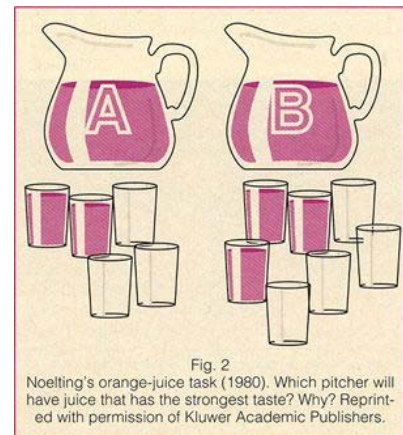


Fig. 2
Noetling's orange-juice task (1980). Which pitcher will have juice that has the strongest taste? Why? Reprinted with permission of Kluwer Academic Publishers.

Proportion problems are also described as being either rate or mixture problems. In *rate* problems, the ratios set up in the proportion are rates—when a change in one unit of one quantity is related to a change in some amount of another quantity. The Mr. Short and Mr. Tall task is an example of this. In *mixture* problems, two quantities are combined to make something new, as in the orange juice task. Tourniaire and Pulos (1985) point out three aspects unique to mixture problems, often making them more difficult: (a) Mixtures create a new entity; (b) students need to understand what happens when the two quantities are mixed; and (c) mixtures generally have the same units.

Solution Strategies. One strategy used by students to solve problems involving proportions is called the *scalar* or *build-up* method. In a missing value problem, a student finds the difference between the numbers in the full ratio, and then scales it up, or down, with repeated addition or subtraction until one of the values in the full ratio matches its corresponding one in the ratio with the missing value. For example, take the problem: *Jane paid 9 cents for 2 sticks of gum, how much did she pay for 8 sticks of gum?* A student might say, 9 for 2, means 18 for 4, means 27 for 6, means 36 for 8. So, the original ratio is

scaled up until the corresponding values match. If you notice, there is *no* multiplication in the problem, rather repeated addition. Piaget, a developmental psychologist who was one of the first researchers to take interest in children's understanding of proportional reasoning, argued that this strategy should be considered "preproportional" because the solution lacks multiplication yet the student recognizes the multiplicative relationship between gum and price.

A second strategy is called the *functional* or *multiplicative* method. Here, the student in this same problem would recognize the relationship between the two numbers in the complete ratio, 9 and 2, and the relationship between the numbers sharing corresponding positions in the two ratios, 2 and 8. Here, the student would notice that to get from 2 to 8 they needed to multiply by 4. Therefore, to find the missing number they must also multiply 9 by 4 to get 36.

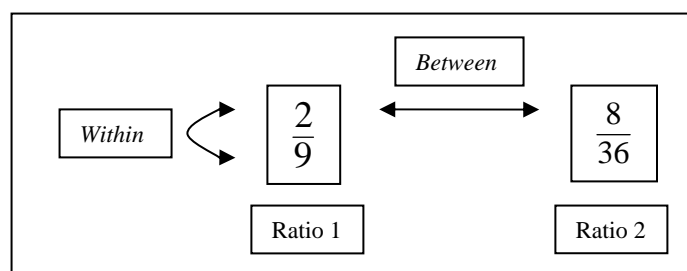
A closely related strategy is called the *unit* method. Here, the student breaks down the complete ratio into a unit rate, and then simply multiplies that rate with the number from the second ratio to find the missing value. In this example, Since 9 cents buys 2 pieces of gum, 4.5 cents would buy 1 piece of gum. Then, you would multiply this *rate* by the 8 to get 36 cents. Notice that the missing value in the second ratio determines which rate to use. Had we known that Jane paid 36 cents, we would want to find the rate of pieces of gum per cent, which in this problem is $2/9$, the multiplicative inverse of 4.5.

The *symbolic* or *cross-multiplication* method is the one often taught in schools. Here, the students set up a written proportion using fractions and an equals sign, $a/b = c/d$. The students fill in the relevant data, cross multiply, and solve for the missing value by dividing. For the previous example, the equation would be $2/9 = 8/x$, so $2x = 9 \cdot 8$, $2x = 72$, divide both sides by 2 and x is 36. Students almost never use this method (MLSC, 2001). Algebra textbooks include an additional method in which a formula for direct proportions ($y = kx$) is used to solve this type of problem. Here, one first uses the equation to find the k —the constant of variation—using the initial information, then solves for k , and finally finds the missing value using the newly created formula. As for the example, one would set x to be gum and y to be price. So $9 = 2k$. Solving for k , we get $k = 4.5$. We now have a function $y = 4.5x$. To find the cost of 8 sticks of gum, one substitutes 8 in for x , getting $y = 4.5 \cdot 8$, simplifying to $y = 36$.

Rates. There are a couple of different definitions for rate described by Lesh, et al. (1988). In both cases, rates are defined in relation to ratios. In one school of thought, rates are believed to compare two quantities in different measurement spaces, whereas ratios describe quantities in the same measurement space. Therefore, a rate would be miles per gallon and a ratio would be like a scale on a map, so many inches on the map correspond with so many inches in the real world. In the second school, rates and ratios are the same, but describe different uses of the same rational number. For example, 3 miles per hour is seen as a rate when it acts as an intensive variable and as a ratio when used to describe the relationship between two quantities, as in for every hour of walking she went three miles. The difference seems subtle, but rates and ratios by this definition have different uses.

Ratios. When describing the types of proportional reasoning problems, there are a couple of additional terms that need to be defined; as they will be used to describe problem difficulty and the methods students use to solve proportions. A *unit ratio* occurs in a problem when the ratio described is 1:n, where n is any integer. An *integer ratio* describes when the ratio of interest is a whole number. For the gum example, the gum to price ratio was not an integer ($9 \div 2 = 4.5$), while the gum to gum ratio was an integer ($36 \div 9 = 4$). The comparisons made by students can be considered within or between (see Figure 3). *Within* refers to the students comparing the ratio given in the problem, and *between* refers to when the students compare across the two ratios given in the problem. For example, a within comparison looks at the ratio of gum to price and a between comparison looks at price to price. The *within* and *between* refer to within and between the ratios set up by the problem. This concept can also be used to describe the ratios in a problem. Hence, in a problem with a *within integer* ratio, the within ratio is an integer, as in 3 pencils for 9 cents. In a problem with a *between integer* ratio, the between ratio is an integer, as in 2 pencils to 8 pencils. Both these ratios can be integers as well, as in the proportion $3/9 = 6/x$, or neither, as in $2/5 = 7/x$.

Figure 3



Student Errors

There are multiple studies, as well as multiple literature reviews of these studies, concerning student thinking and performance on proportional reasoning tasks, most often on comparison and missing value problems. In this section, we will describe some of the pooled findings and then specific studies and their results. In the next section, we will look at some of the elements of proportion problems that affect student proficiency, as well as the methods students use.

Error Strategies

Tournaire and Pulos (1985) summarize the main error strategies employed by students on proportion problems from a litany of studies:

1. *Students ignore part of the question or some of the data.* On a missing value problem, a student uses only two of the three pieces of given information to find a solution. Using the gum example, a student might say 8 sticks of gum costs 72 cents since 9 cents times 8 sticks is 72, not using the piece of data which tells us

that the 9 cents is for 2 sticks of gum.

2. *Students use an additive strategy that focuses on the constant difference between two numbers in a proportion*, as opposed to their constant rate. This strategy is called *constant difference*. Here, a student might notice that 9 cents is 7 more than 2 sticks, so to find the missing value they would add 7 to 8, the number of sticks of gum in the second ratio, to predict that 8 sticks would cost 15 cents.
3. *Students may use a combination of an additive and multiplicative strategy*, often occurring with noninteger ratios. Here, a student may find the correct noninteger ratio, but only multiplies by the whole number component of the ratio to find the missing value, adding the remainder. For the gum problem, a student might say that $9/2$ is 4 R1, so they would multiply 8 by 4 and then add 1, getting 33 instead of 36.
4. *Students may also use a faulty application of a correct strategy*. They may set up a proportion, but put the numbers in the wrong places. They may find the wrong unit rate, but use it correctly to find the missing value.

Karplus and Peterson (1970), and many more studies by Robert Karplus and others throughout the 1970s, categorized student responses while working on the Mr. Short and Mr. Tall task. They coded six error strategies that align but do not match those of Tourniaire and Pulos (1985):

- N. *No explanation*. The student did not respond to the question.
- I. *Intuition*. The student made a decision based on the appearances of other extraneous factors in a problem, often called using a *qualitative method*. A student may say that they feel one ratio is larger than the other because they like the number 5, and not because of the values of the two ratios.
- IC. *Intuitive computation*. The student makes use of some of the data and combines it haphazardly, as in the first category described above.
- A. *Addition*. This matches the *constant differences* method
- S. *Scaling*. The student makes a change in scale that they predict given information not given in the problem. For example, in the Mr. Short and Mr. Tall problem, the student may assume that the little paper clips are half the size of the big paper clips. Therefore, they choose a scale factor of two and make a prediction accordingly. In actuality, the scale was only 1.5 for the task (remember, $4/6 = 6/x$).
- AS. *Add and Scale*. This matches additive and multiplicative strategy.

In a later experiment, Karplus, Karplus, and Wollman (1974) changed the task to use buttons and small paper clips. By eliminating the large paper clips, the students were no longer able to fixate on the relationship between the sizes of the paper clips. Although all of the numbers were left the same ($4/6=6/x$), they found almost no uses of the scaling strategy! In other words, the way a problem is presented will affect how students interact with it.

Why Additive?

The constant difference or additive method is by far the most common erroneous strategy used by students when working with proportion tasks. Here, the students look at the difference between the two numbers in the within a ratio, instead of their quotient, to make a prediction on the missing value in the second proportion. For example, in the paint example above ($3/6=7/x$), the students would notice the difference between 3 and 6 is 3, and would then guess that x is 10, $7 + 3$. This method can also be used in a comparison task, where the students would compare the differences, and not the ratios, or the paired numbers. The rational numbers tool has additional information on comparison of rational numbers.

Not all additive methods result in failure. The build-up method can be effective. This is where students use successive addition to scale one ratio up (or down) so that it matches the other in some way, for either comparison or missing value tasks. In the paint example, a student might double 3 yellow and 6 red to get 6 yellow and 12 red. They need to get to 7 yellow, so a more advanced student might see that they only need to scale up 3 and 6 by one third, 1 and 2, to get 7 yellow and 14 red. This method is often more successful when the between ratio is an integer (i.e., $3/7 = 9/x$, students would scale up twice, 3:7 become 6:14 becomes 9 to 21, so the answer is 21).

Hart (1984) believes that one of the reasons that students refuse to leave additive methods is because they have success with them on easier problems. As shown by the two examples in the previous paragraph, repeated addition works easily with the second set of numbers, but is more difficult in the first. Hart suggests that since students work on the easier problems first, and find success, the students believe that their strategies will work with any proportion problem.

Wollman and Karplus (1977) give three reasons why the additive strategies are stable. First, addition is the arithmetic operation most understood and familiar to students. Second, when multiplication and division are originally taught, they are learned as repeated addition and division. Last, remainders from division are treated as additive items.

A number of studies found that students resorted back to additive strategies when problems became more complex—using noninteger ratios. Karplus, Pulos, and Stage (1983a) found that students working on *comparison* problems used proportional reasoning to solve problems where the ratios were equivalent and the within ratios were integers. The students used more additive strategies when the between ratios were integers, neither ratios were integers, and when the two ratios were not equivalent. In Hart's (1984) study, the students used repeated addition correctly on problems with integer ratios, but then used constant difference methods when the ratios became more difficult. Tourniaire and Pulos (1985), in their review of proportional reasoning studies, suggest that students “fall back” on elementary strategies, like addition, on more difficult problems.

The preponderance of additive solutions is further complicated by how little students use cross-multiplication, the textbook method that they are taught in school. Lesh, Post, and

Behr (1988) believe this is the case because the method is poorly understood by students, it is seldom a naturally generated solution, and when it is taught, it is done so in a way to avoid proportional reasoning instead of facilitating it. Cross-multiplication is taught as a procedure that is not understood. Students are told to put numbers into a formula and compute. Rather than being a tool that is used as a result of proportional reasoning to find a solution, helping to create a generalized algorithm that works with all missing value tasks, students are taught cross-multiplication in lieu of proportional reasoning.

The Math Learning Study Committee (2001) holds that “many school mathematics programs fail to develop children’s understanding of ratio comparisons and move directly to formal procedures for solving missing-value proportion problems” (p. 417). They also believe that the presentation of cross-multiplication before students understand proportional relationships is one of the causes for its nonuse. Behr, Harel, Post, and Lesh (1993) believe that elementary-school mathematics is partly to blame for students difficulty with proportions because it fails to include the basic concepts and principles related to multiplicative structures needed to work with ratios.

Lastly, Wollman and Karplus (1977) believe that part of the problem is that ratios are introduced as fractions and proportions as equivalent fractions in school textbooks. They believe that math curricula do little to interpret ratio as a correspondence of measures. To gain a better understanding of the structures underlying proportional relationships, *they believe that students should work with physical representations of ratios and proportions before learning how to solve these types of problems symbolically at a more abstract level.*

Teachers Conceptions of Students

Several studies look a different part of the problem: that teachers are more inclined to use formulas and algebra to solve proportions while, as we have seen, students naturally use more intuitive and informal strategies. In a study of teacher understanding of proportions, Fisher (1988) found that though teachers were successful on straightforward direct proportion tasks, they achieved less than 50% on inverse proportions (which are not taught until high school, and thus not include in this tool), and around 70% on more complex direct proportion problems. She found that teachers almost always make use of formulas and algebra, and do little reasoning. In other words, the less formal strategies used by students are not used by teachers. Therefore, Fisher discussed that when the teachers were asked how they would teach such problems to their students, very few discussed the less formal strategies that students might use.

Misailidou and Williams (2002) found that nine preservice teachers were excellent at predicting the difficulty level of proportional reasoning problems for their students, but were not good at predicting their correct and error strategies. The teachers often thought the students would use cross-multiplication, or simple multiplication with unit ratios, to get correct answers, and were often unable to guess how the students would make mistakes. These teachers also did not know that pictures and diagrams would be, and are an appropriate tool to improve student achievement with proportions.

Correct Strategies

Wollman and Karplus (1977), in a study of 450 suburban 7th and 8th graders, found that 1/5 of the students they tested used proportional reasoning consistently across all of the problems and 1/4 used proportional reasoning on some of the problems. Some of the studies look at what these successful “proportional reasoning” methods look like. The three most common correct strategies in the literature, according to Tourniaire and Pulos (1985), are multiplication, build-up/repeated addition, and the infrequently used cross-multiplication.

Correct strategies can be further broken down. For example, Weinberg (2002) elaborates on some effectively used strategies. She had 387 students complete a missing value problem on a map scale: 5 miles is to 9 cm, what is 2 miles? Of the 90 students who got the correct answer, 43 found a unit rate (for every mile there are 1.8 cm, so 2 times 1.8 is 3.6), 3 did repeated subtraction (5 to 9 is 4 to 7.2 is 3 to 5.4 is 2 to 3.6), 16 used division and multiplication (if 2 is 2/5 of 5, than what is 2/5 of 9, 3.6), and 27 used cross-multiplication (set up $5/9 = 2/x$, and gets $5x = 18$, and solves for x). Karplus, Karplus, and Wollman (1974) broke down proportional reasoning into three categories: students who used repeated addition, students who created and compared ratios, and students who compared the magnitudes of the physical characteristics of the objects in a problem (those who figured out a button was 1.5 paper clips).

Difficulty Factors

Many of the studies list variables that influence student achievement on proportion tasks. As taken from Tourniaire and Pulos (1985), we divide these variables into three categories. *Structural variables* refer to the structure of the numbers and the ratios in the problem, *context variables* concern how the problem is presented, and *student variables* looks at the students rather than the structure or context of the problems they are working on.

Structural Factors

The structural variables of proportion problems include integer ratios, the order of the missing number, the size of the numbers, use of unit ratios, and, in comparison problems, whether the ratios are equivalent. Research has found that, for students, 1:2 ratios are the easiest, then 1:n (this is a unit ratio, where n is any integer), then nonunit. Finally, noninteger ratios are the most difficult. Furthermore, it has been found that the more difficult proportion problems included the lack of integer ratios, a missing value that is smaller than the original ratio, and when the numbers in the proportion get larger than 30 (Tourniaire & Pulos, 1985; Noetling, 1980a; Ripley, 1981).

There also has been some debate as to the comparison of between and within ratios. Problems with within-integer ratios appear to be easier than those with only between-integer ratios. Karplus, Pulos, and Stage (1983a), found that students made much more frequent use of within comparison than between. Therefore, problems with within-integer ratios would be easier than problems with only between-integer ratios. Students were able

to answer problems with within- integer ratios at a significantly higher rate than those with between-integer ratios. They found, as before, that harder problems elicited more additive errors. Although research indicates that students have more difficulty with problems that do not have an integer between ratio, Karplus, Pulos, and Stage (1983b) have found that students are affected by how the problem is set up. They suggest that students are sensitive to not only the magnitudes of the numbers used, but also the meaning of the data.

Problem Context

Tourniaire and Pulos (1985) summarize context variables as follows. They found that mixture problems were more difficult than rate problems. Although discrete quantities are easier to visualize than continuous ones (i.e., nails versus pounds of nails), suggesting that problems with discrete quantities may be easier, the findings have been mixed. *The familiarity of the students with the context of the problem, and the availability of manipulative and visual representations accompanying problems significantly improve student performance.* Misailidou and Williams (2002) found that students improved from 17% to 55% with the inclusion of a picture on the paint-mixture problem from before. Wollman and Karplus (1977) were able to compare methods and context in four different tasks. On the Mr. Short and Mr. Tall task, they found that 32% of the students used additive and 40% used proportional strategies. On a candy task, which was given verbally without manipulatives, 50% of the students used additive strategies, while 26% used proportional ones. On a ruler task that was both familiar and concrete, the students used proportional strategies 87% of the time, and additive only 2%. Last, on a task with pulleys that was concrete but more difficult to comprehend, the results were identical to the Mr. Short and Mr. Tall task. *Familiarity with the task and physical representations improved student success.*

Karplus and Peterson (1970) devised the Mr. Short and Mr. Tall task with different sized paper clips and found that a large number of students used what they called a scalar methods, where the students assumed the paper clips were in a ratio of 1:2 (it was actually 2:3), and made a multiplicative decision based on the appears of the paper clips and not the data presented. Karplus, Karplus, and Wollman (1974) changed the task, having buttons replace the large paper clips. The scalar method disappeared, nearly completely. Their conclusion is that the presentation of the problem affects both the method the students employ and their proficiency. They hold that the new presentation “compels” the students to make use of the number data in the problem, rather than their perception of the objects used to measure. Wollman and Karplus (1977) found that on written computation tasks not placed in word problems, students fared much better on ones depicted geometrically, comparing sides of similar shapes, rather than those presented numerically. Successful performance on geometric tasks did not necessarily indicate success on numerical ones, but the opposite did hold.

Student Characteristics

Individual characteristics of the students working on these problems are split into two categories: psychological and socio-economic. Tourniaire and Pulos (1985) describe the

former. First, older students are better than young students at proportional reasoning. Second, there are mixed findings on sex/gender, however when a difference is found in the literature, males perform better than females. Third, a positive attitude towards mathematics seems to affect the performance of students on more difficult proportion tasks. Fourth, a construct called *M-capacity*, or the ability to work with multiple pieces of information at once, is positively correlated with performance. *Since proportion tasks involve four numbers, two quantifiers, two ratios, and an equality, there is a lot of information that students have to work with simultaneously to correctly solve these problems.* Finally, a construct called *field-dependence* also influences performance. This describes a person's ability to resist irrelevant clues and strategies. A student who is field-independent can decide which information is pertinent to a problem, and the correct strategy to get through that task. Tourniaire (1986) found significant correlations between performance on proportion tasks and M-capacity, field-dependence, and general intelligence. In other words, the more the M-capacity, the more field-independent, and the higher the IQ, the more successful the students were.

Karplus, Pulos, and Stage (1983a) hold that while upper-class and upper-tracked students appear to use proportional reasoning by age 12, only a small fraction of low-income and/or lower-tracked students use it by ages 14 to 17. These findings come from Karplus and Peterson (1970). Here, the original Mr. Short and Mr. Tall task was given to 727 students in grades 6, 8-10, and 11-12 from both suburban and urban schools. They found no statistical difference between the performances of the students at the 6th grade level, but the differences became significant at 8-10, and disturbing by 11-12. In grades 8-10, students in the suburbs used proportional strategies 32% compared with 5% or urban students. By grades 11 and 12, this had shifted to 80% of suburban students compared with only 9% of urban students using these strategies.

Instructional Strategies

Tourniaire and Pulos (1985) advise educators that proportional reasoning is not acquired quickly. Rather, students first use elementary strategies to solve these sorts of problems, and hold on to these strategies, especially additive ones, long after they have been effective. Lastly, they suggest that “comparing ratios is an advanced method, and the ability to choose the arithmetically easier comparison is acquired long after the proportional techniques are mastered” (p. 188).

The Math Learning Science Committee (2001) proposes three skills, that once learned, improve student proficiency on proportional reasoning tasks. First, students must learn to make comparisons based on multiplication rather than just those based on addition. Second, students need to recognize which aspects of a proportion can change, and which remain the same. Third, and last, students need to learn to build composite units. Here, the authors are referring to the ability to see a ratio as two separate extensive quantities (5 miles and 2 hours) and as one intensive quantity ($5/2$ miles per hour).

According to Tourniaire (1986), success on proportional tasks is based on four abilities. First, and foremost, is the student's ability to draw on and utilize an “appropriate scheme”

for a problem. The scheme represents the student's understanding of both a type of problem and a successful strategy for solving that type of problem. The other three abilities we have seen before: The student's M-capacity, or ability to work with multiple pieces of information simultaneously; the student's ability to use their full M-capacity, and their field-dependence.

In the classroom, Tourniaire and Pulos (1985) state that to improve proportional reasoning, the field suggests the following:

1. Lab activities should be provided for students to get a hands-on and concrete experience with proportions to serve as a basis for proportional reasoning.
2. Students' attention should be directed to the relevant variables in a problem, including the numbers and the descriptive words that indicate an additive or multiplicative relationship.
3. Students should get practice distinguishing between additive and multiplicative solution strategies.
4. Students should practice using different correct solution strategies.

Post, Cramer, Behr, Lesh, and Harel (1993) believe that proportion tasks are taught divorced from their meaning. Instead of analyzing the multiplicative relationship between the numbers in a proportion, students are often taught to place the numbers into a proportion and cross-multiply. Here, *the symbolic work acts as a way to get around proportional reasoning*—students memorize procedures that they do not understand. They also hold that students have difficulty getting beyond whole numbers. This is why they believe students who multiply to solve proportions with integer ratios use addition to solve proportions with noninteger ratios. When dealing with nonwhole numbers, they resort to a method that only uses whole numbers. They make the following suggestions (p. 344):

1. Extend interpretations of rational numbers and develop connections among them. Instruction should build on previous learning and understanding should be expected to evolve over a time period of several years.
2. Instruction should emphasize the interrelationships within the rational number domain (part-whole, decimal, ratio, measure, and operator).
3. Procedures and operations should be delayed until an understanding of quantities is established. Understanding of quantities should include an emphasis on order and equivalence ideas.
4. Develop understandings via instructional models that reinforce links between concepts and procedures as well as translations within and between modes of representations.

Hart (1984) suggests that teachers can avoid frequently used error methods if they are first aware of, and accommodating to, student methods. Simply showing students how to set up a proportion and then cross-multiply is no guarantee that those students will use that method. Instead, student-generated methods can be used as a springboard into understanding proportional relationships. Fisher (1988) suggests that students' informal strategies can be used to create a better conceptual understanding of proportions. Here, students would look at the mathematical relationships between the numbers in a proportion and what successful solution strategies look like, rather than learning a procedure without understanding its underlying structure. Fisher also holds that using students' methods would support generalization.

Hart (1984) created a teaching module that "eradicated" student errors on proportion tasks on both a post-test and a delayed post-test. The teaching module included four subcomponents:

1. *Conflict*. The teacher demonstrates how students' additive methods fail when problems become more difficult.
2. *Times*. Students are taught that multiplication is the correct operation for completing missing value and comparison tasks.
3. *Fractions*. Students are retaught how to multiply fractions. Hart believes that one of the reasons students use additive strategies on more difficult problems is because they do not want to multiply fractions. Hart found that students still had difficulty with multiplying fractions after reteaching them in earlier versions of the module, so she replaced this component with one teaching students how to use calculators to multiply fractions.
4. *Scale factor*. Students are given a method for finding the scale factor in a problem (unit rate).

Kurtz and Karplus (1979) did a study using the *Numerical Relationships Program* of Kurtz (1976). The program took 14 50-minute sessions and saw substantial results, including the use of algebra to solve proportion tasks on a post-test, and multiplication on a delayed post-test. The program included six differences from traditional textbook methods of teaching proportional reasoning:

1. Uses lab activities;
2. Follows a cycle of exploration, concept introduction, and concept application;
3. Focuses attention on appropriate variables within a given task;
4. Offers students practice distinguishing between constant ratio, constant difference, and constant sum situations;
5. Develops proportional reasoning from the tendency of students to use the build-up/repeated addition strategy informally; and
6. Avoids algorithmic techniques.

An important finding here, and with all of the studies, is that these suggestions do not work with every student. In their study, they found that 50% improved their skills with the intervention, while 16% already used proportional reasoning prior to the program. That leaves 33% who did not change.

Ben-Chaim, Fey, Fitzgerald, Benedetto, and Miller (1998) found a similar finding when comparing a *standard-based* curriculum, Connected Mathematics Project (CMP), and a traditional one. The standards here refer to the standards documents of the NCTM (2000), most notably the *Principles and Standards for School Mathematics*. According to Ben-Chaim, et al., the CMP approach is to “encourage students to construct their own procedures for doing rational number computations, solving proportions and applying those skills to applied problem solving,” and to do so, “students collaborate in work on the problems, having their diverse insights and approaches with partners and then with the whole class through mathematical reflections, discussions, and journal writing” (p. 248). As for their findings, the CMP curriculum was shown to be significantly more effective than the traditional one, with 53% of students giving correct answers, with correct work, on four missing value and one comparison task as opposed to only 28% of students with the traditional curriculum. However, at 53% there is still room for improvement. An additional finding was that the CMP curriculum fostered “good quality” oral explanations from students. In other words, having the students work in a collaborative environment to gain a deeper understanding of proportional reasoning will improve their ability to talk about the strategies they use to solve these problems.

Conclusion

The ability to work effectively with proportions is clearly associated with students’ understandings of rational numbers and multiplicative relationships. The literature suggests that proportional reasoning is difficult for students. The same literature has also included ideas about how to improve student performance with proportional reasoning tasks: *The use of diagrams, physical representations, familiar contexts, connecting intuitive methods with more formal algebraic ones, delaying the teaching of cross-multiplication, etc.*

In a way, it is appropriate to end this summary of the work over the past 40 years on proportional reasoning with the conclusion from one of the first studies. In the words of Karplus and Peterson (1970), “it may be disappointing that successful proportional reasoning is not achieved earlier than the last years of high school, even though the subjects of ratio and proportion make their appearance in most mathematics programs in junior high school. ... It seems, therefore, that there is a serious gap between secondary school mathematics and science curricula and the students’ reasoning ability” (p. 818). In the end, proportional reasoning is a difficult topic that involves students using a lot of numerical and contextual information, simultaneously, to solve a problem based on a number system that they do not always well understand.

References

- Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *NCTM handbook for research on mathematics teaching and learning* (pp. 296-333). Reston, VA: National Council of Teachers of Mathematics.
- Ben-Chaim, D., Fey, J., Fitzgerald, W., Benedetto, C., & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics*, 36(3), 247-273.
- Fisher, L. C. (1988). Strategies used by secondary mathematics teachers to solve proportion problems. *Journal for Research in Mathematics Education*, 19(2), 157-168.
- Hart, K. M. (1984). *Ratio: Children's strategies and errors. A report of the strategies and errors in secondary mathematics project*. London: NFER-Nelson.
- Karplus, R., & Karplus, E. (1972). Intellectual development beyond elementary school III—Ratio: A longitudinal study. *School Science and Mathematics*, 72(8), 735-742.
- Karplus, R., Karplus, E., & Wollman, W. (1974). Intellectual development beyond elementary school IV: Ratio, the influence of cognitive style. *School Science and Mathematics*, 74, 746-482.
- Karplus, R., & Peterson, R. (1970). Intellectual development beyond elementary school II: Ratio, a survey. *School Science and Mathematics*, 70, 735-742.
- Karplus, R., Pulos, S., & Stage, E. K. (1983a). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (eds.), *Acquisition of mathematics concepts and processes* (pp. 45-90). Orlando, FL: Academic Press.
- Karplus, R., Pulos, S., & Stage, E. K. (1983b). Early adolescents' proportional reasoning on 'rate' problems. *Educational Studies in Mathematics*, 14, 219-233.
- Kurtz, B., & Karplus, R. (1979). Intellectual development beyond elementary school VII: Teaching for proportional reasoning. *School Science and Mathematics*, 77, 387-397.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*, (pp. 629-666). Reston, VA: National Council of Teachers of Mathematics.
- Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: Lawrence Erlbaum Associates & National Council of Teachers of Mathematics.

- Math Learning Study Committee (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Misailidou, C., & Williams, J. (2002). "Ratio": Raising teachers' awareness of children's thinking. Paper presented at the 2nd International Conference on the Teaching of Mathematics, Hersonissos, Greece.
- New Jersey Department of Education (2004). *New Jersey core curriculum content standards*. Retrieved June 2, 2006, from http://www.state.nj.us/njded/cccs/s4_math.pdf.
- Noetling, G. (1980a). The development of proportional reasoning and the ratio concept: Part I – Differentiation of stages. *Educational Studies in Mathematics*, 11, 217-253.
- Noetling, G. (1980b). The development of proportional reasoning and the ratio concept: Part II – Problem structure at successive stages; problem solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics*, 11, 331-363.
- Post, T. R., Cramer, K. A., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching and assessing of rational number concepts. In T. Carpenter, E. Fennema, & R. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-362). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics.
- Texas Education Agency (1997). *Chapter 111. Texas essential knowledge and skills for mathematics, Subchapter B. Middle school*. Retrieved June 2, 2006, from <http://www.tea.state.tx.us/rules/tac/chapter111/ch111b.html>.
- Tourniaire, F. (1986). Proportions in elementary school. *Educational Studies in Mathematics*, 17, 401-412.
- Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics*, 16, 181-204.
- Weinberg, S. L. (2002). Proportional reasoning: One problem, many solutions! In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 138-144). Reston, VA: National Council of Teachers of Mathematics.
- Wollman, W., & Karplus, R. (1977). Intellectual development beyond elementary school V: Using ratio in differing tasks. *School Science and Mathematics*, 77, 593-611.