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Rational Numbers: Fractions, Decimals, and Percents
Student Misconceptions and Strategies for Teaching

Rational numbers present significant challenges for many students. In fact, many of the researchers on the teaching of rational numbers come to similar conclusions about the misconceptions and problems that students have (Behr, Wachsmuth, & Post, 1995; Hiebert & Wearne, 1986; Lindquist, Carpenter, Silver, & Matthews, 1993):

- Many students understand the procedures involved in working through problems with rational numbers, but do not have a rich, conceptual understanding of what these numbers mean.

- Much instruction focuses on syntactic (procedural) rather than semantic (conceptual) rules for rational numbers.

- Many students confuse the rules for manipulating whole numbers when working with rational numbers.

- Many students have trouble using models to illustrate operations or to connect operations on objects to algorithms.

In the domain of fractions, for example, the majority of ninth-grade students, when asked to estimate the sum of 11/12 + 7/8 chose 19 or 20 as the answer in a multiple-choice test (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1980). When students were asked to select the largest of four decimal numbers: .09, 385, .3, and .1814, none of the students in fifth grade selected the correct answer, while only 43% of those students tested in ninth grade selected the right answer (Brown, 1981; Carpenter, et al., 1981; Ekenstam, 1977; Grossman, 1983). In the work of Lembke and Reys (1994), it becomes clear that not all students understand 100 as the base for working in percents. Indeed, many middle-school students do not understand conceptually what rational numbers mean.

State Standards

Both New Jersey and Texas clearly define what students should be expected to accomplish by the end of each grade in middle school. As the grades progress, students are expected to complete more difficult tasks associated with each standard. The standards (NJDE, 2004; TEA, 1997) can be reduced to five main areas of expertise:

1. Students should be able to understand rational numbers and relate them to real-life situations.

2. Students should be able to switch between equivalent forms of rational numbers including whole numbers, fractions, decimals, and percents.

3. Students should be able to order positive rational numbers from least to greatest and vice versa.

4. Students should be able to add, subtract, multiply, and divide whole numbers, fractions, and decimals. They should be able to explain how to do so with paper-and-pencil, with mental math, and with a calculator.

5. Students should be able to use proportional reasoning to solve problems involving rates and ratios.

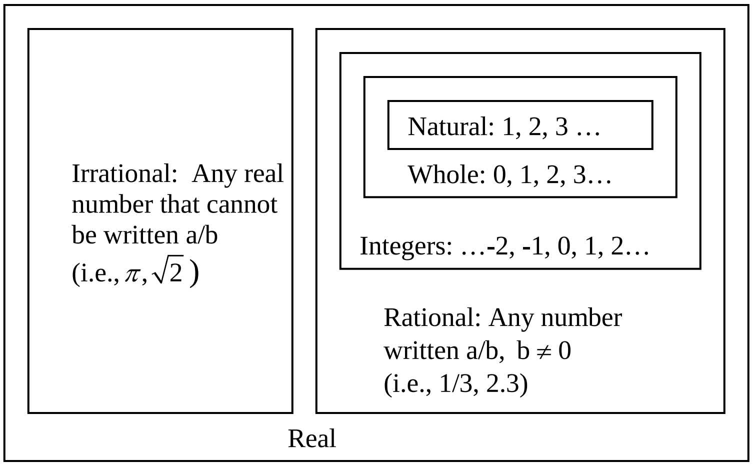
What Do Students Need to Understand in Working With Rational Numbers?

What Are Rational Numbers?

Rational numbers typically are defined as follows:

> The set of numbers that include fractions and decimals. Formally, the definition often states that they are the set of numbers that can be written as "a over b" where a and b are integers [i.e., whole numbers and their opposites] and b is not equal to zero. In terms of a number line, the rational numbers exist in the spaces between the integers, as well as including the integers themselves. (Ridener & Fritzer, 2004, p. 7)

For the purposes of this discussion, numbers in the form of percent are also included, although most of the research does focus on fractions and decimals.



Another way to understand rational numbers is in their relation to other number systems. The most basic number system is composed of the *natural numbers* or *counting numbers*, which are whole numbers starting with 1 and going to infinity. The *whole numbers* are a larger set by one, containing the more recently discovered number 0.

If you take all of the whole numbers and include all of their opposites—in other words all of the negative whole numbers— you get a group called the *integers*. The next larger set is the *rational numbers*, whose definition is above. The next set is not a subset of rational numbers, but rather a rival set. Any decimal that does not repeat and is never ending, i.e. any number that is not a rational number, is considered an *irrational number*. Amazingly, this set is infinitesimally larger than that of rational numbers. Between any two rational numbers there is an infinite amount of irrational numbers. Lastly, for this discussion, we have *real numbers,* which are composed of both rational and irrational numbers.

While students may not fully understand the formal definition given above, they do have informal notions related to rational numbers before they enter middle school. At an early age, children develop a basic understanding of the concept of "fair shares," dividing something evenly among a given number of people or objects (Kilpatrick, Swafford, & Findell, 2001). Children also develop intuitive understanding of proportionality in simple settings; they are capable of identifying a glass as full, empty or half-empty, and can make relative evaluations of about which of two glasses is "more full" (Moss, 2002). Finally, students develop simple multiplicative strategies that pave the way to a fuller understanding of fractions. Specifically, research suggests that children develop the capacity to "split" a given quantity in half, either as means of doubling its number or halving its size. This process allows students to manipulate numbers in a manner quite different from their more fully developed capacity to add and subtract (Confrey, 1994).

The Ability to Translate Fractions, Decimals, and Percents

Students require a better conceptual understanding of numbers. Traditionally, mathematics instruction has focused more on the procedures involved in solving a problem, without first spending time on developing an understanding of what rational numbers represent.

To understand rational numbers, students must be able to make sense of their various meanings and notations. For example, 1/4 can be understood as both a fraction of a whole and as a division problem (1 divided by 4). Further, students must be able to convert rational numbers from one representation to another (e.g., fraction to decimal, or decimal to percent) and to make comparisons between them. A rational number can be interpreted in five different ways:

> Research over the past two decades identifies the following interpretations for any rational number, say 3/4: (a) a part-whole relation (3 out of 4 equal-sized shares); (b) a quotient (3 divided by 4); (c) a measure (3/4 of the way from the beginning of the unit to the end); (d) a ratio (3 red cars for every 4 green cars); and (e) an operation that enlarges or reduces the size of something (3/4 of 12). The task for students is to recognize these distinctions and, at the same time, to construct relations among them that generate a coherent concept of rational number. (Kilpatrick, Swafford, & Findell, 2001, p. 233)

Proportional Reasoning Skills

In order to make sense of rational numbers and forge connections between their various representations, it is critical that students develop proportional reasoning skills. These skills, which include an ability to apply concepts of ratio and proportion to problem-solving tasks, are often embedded in students' understanding of rational-number concepts, and are a key component of some instructional strategies to develop those understandings. For example, students' ability to halve or double a given quantity—say, 50%—allows them to move from rational numbers that are intuitive to those that are abstract. Students may not intuitively understand the fraction 1/8, or 12.5%, but through proportional reasoning they are able to develop an understanding of these numbers by working down from 50%, or one-half (Moss, 2002).

The concept of proportion is based on the relationship between ratios, which themselves express a fixed, multiplicative relationship between two quantities. For example, if 3 gallons of gasoline costs $8.00, the ratio of gallons to dollars is 3:8. A proportion allows the specific values to vary while maintaining a consistent ratio. In the case of the gasoline example, a proportion indicates that paying $24.00 for 9 gallons of gas is the same relationship as paying $8.00 for 3 gallons.

Three critical elements anchor students' understanding of proportional reasoning.

- First, students must move from additive to multiplicative reasoning as they learn to discern between *absolute change* between two values (e.g., the number of inches a child grows in one year) and *relative change* (the amount the child grows relative to her previous height).

- Second, students must understand that while the specific values within a proportional relationship can change, the relationship between the values must remain constant.

- Third, students must be able to construct a unit scale based on a given proportion in order to effectively manipulate the values while maintaining the proportional relationship. In the example given above, students must understand that if 3 gallons of gasoline costs $8.00, the cost per gallon (the unit cost) of gasoline is approximately $2.67 per gallon (Kilpatrick, Swafford, & Findell, 2001).

Fractions

Orton, et al., (1995) describe three characteristics of a child's ability to understand what fractions mean:

- The first characteristic is a child's ability to translate between different expressions of fractions. For example, the rational number 2/3 can be expressed symbolically as 2/3 or as a circular region cut into three parts with two of these parts shaded. The child who “understands” rational-number concepts can make translations between these different embodiments or representations of the fraction concept.

- A second characteristic is that a child can translate among different ways to express a concept within the “same form” of representation. For example, if a child is using chips to represent fractions, she can recognize that 4/6 is equivalent to 2/3 by restructuring an array of four black chips and two white chips (representing 4/6) into an array of two equal sets of black chips and one equal set of white chips (representing 2/3).

- The third characteristic is the child’s ability to progressively move away from concrete representations of rational numbers to more symbolic modes. For example, when asked to compare 5/6 and 2/3, a child might devise a plan without actually using the physical chips. This planning without the manipulative anticipates a more formal understanding of number magnitude. (Orton, et al., 1995).

When comparing fractions, Cramer, Behr, Post, and Lesh (1997a, 1997b) named four successful strategies noted in students’ thinking that involve some conceptual understanding: same numerator, same denominator, transitive, and residual strategies.

- **Same Numerator:** When comparing 2/3 and 2/6 (fractions with the same numerator), students can conclude that 2/3 is the larger fraction because thirds are larger than sixths and two of the larger pieces must be more than two of the smaller pieces. This strategy involves understanding that an inverse relationship exists between the number of parts a unit is partitioned into and the size of the parts.

- **Same Denominator:** This strategy refers to fractions like 3/8 and 2/8. In this case, the same denominator implies that one is comparing parts of the unit that are the same size. Three of the same-size parts are greater than two of the same-size parts.

- **Transitive:** This strategy can be modeled by comparing 3/7 and 5/9. When making this comparison, a student can conclude that 3/7 is less than 5/9 because 3/7 is less than 1/2, while 5/9 is greater than 1/2. This is the transitive strategy because students use a single outside value to compare both fractions.

- **Residual:** When comparing 3/4 and 5/6, a student can reflect that both fractions are one “piece” away from the whole unit. Since 1/6 is less than 1/4, 5/6 must be closer to the whole and is therefore the bigger fraction. This thinking strategy has been called a *residual* strategy because students focus on the part “leftover” in judging the relative size of the fractions.

Percents

Lembke and Reys (1994) identify five components of understanding percents:

- Development of a pictorial representation for percents.
- The ability to relate fractions, decimals, and percents.
- The ability to apply benchmarks and other estimation techniques to percent problems.
- The ability to perform mental computations in conjunction with percent problems.
- A sensitivity to the reasonableness of an answer.

Their research has found each of these concepts of understanding to be important. The use of benchmarks was found to be especially helpful; this understanding can be developed with students as they grow older, starting with only rudimentary benchmarks such as 50% and 100% initially. The ability to relate fractions and decimals to percents is also crucial. Moreover, it is important for students to understand and appreciate the usefulness of percent in making comparisons, given that percents have a common reference point (out of 100).

However, students first must have a solid understanding that all percent problems start with 100 as the base. Without this essential understanding, students will have difficulty with most computational problems. Instruction also can use the informal knowledge with which students come to the classroom. They see percents as a function of daily life, and this experience can be used in teaching.

What Problems and Common Misconceptions Do Students Have About Rational Numbers?

Confusion With Whole-Number Rules

Some of the properties of whole numbers are different from those of rational numbers, and even though the rules governing the operations of rational numbers are identical to those of whole numbers, some of the common practices and vocabulary used for operation on whole numbers do not hold for rational numbers. These two issues make working with fractions and decimals seem counterintuitive in some cases. As they learn to count, students develop a strong intuitive understanding of whole numbers. Vambakoussi and Vosniadou (2004) suggest that “in mathematics education research, there has been much evidence to show that prior knowledge about natural numbers stands in the way of understanding rational numbers. Students make use of their knowledge of whole numbers to interpret new information about rational numbers” (p. 456). These understandings can be misleading when applied to rational numbers, and misconceptions can be compounded as students practice their use of algorithms without sufficient knowledge of the underlying concepts of whole-number operations. For example:

- Students often think that .18 is greater than .9 because on the whole-number scale, 18 is larger than 9. Similarly, students may read 1/4 as larger than 1/3 because 4 is a “bigger” number than 3 (Kilpatrick, Swafford, & Findell, 2001).

- As students learn to compute with whole numbers, other conceptions arise which make understanding rational numbers problematic. In learning addition, students often learn to line up numbers “on the right” to ensure that place values are aligned (ones, tens, hundreds, etc.). When adding decimals, this gives rise to simple errors, such as students’ thinking that the sum of .42 and .7 is .49 (Behr, Wachsmuth, Post, & Lesh, 1984). Another common assumption from whole-number operations is that when a number is multiplied, it gets larger, and when divided, it gets smaller. Obviously, the reverse can also be true when working with numbers smaller than one (Fischbein, et al., 1985).

- One of the problems in the learning rational numbers is that whole-number counting strategies fail to work (Behr & Post, 1988). With whole numbers, the idea of counting the “next number” makes sense. The next number after 4 is 5. This idea does not make sense in the rational numbers, which makes rational numbers difficult to count. For example, there is no next fraction after 1/2. Put formally, the rational numbers are “dense,” meaning that between any two rational numbers there is always at least one (actually an infinite number) rational number.

- Additionally, Vamvakoussi and Vosniadou (2004) found that students often had difficulty deciding on the amount of numbers between any two rational numbers. Because of student reliance on the order of the whole numbers, it was difficult for them to understand the density of rational numbers. Asked how many rational numbers are between 2.3 and 2.4, a majority of children said none or 9, while very few said infinity. As for the response of 9, these children copied a whole-number principle: 2.31, 2.32, 2.33, up to 2.39. These students could not repeat this simple heuristic to prove that there is an infinite amount of numbers between any two rational numbers.

- Students often learn about only a small subset of rational numbers—the rational numbers between zero and 1. This leads to problems in understanding improper fractions, which represent numbers larger than 1. In some cases, students seem to think that any fraction greater than 1 is equal to 1, while in other cases students find nonfraction representations for their answer (Moskal & Magone, 2002).

The conflation of rational and whole-number concepts may be exacerbated by certain instructional approaches. This is particularly true in cases where a “set model” is introduced to explain fractions. Sets of objects are often used to show fractions as a part of a whole. Students, however, do not naturally equate the set with the whole, instead seeing it as multiple items. Some research has suggested that this confusion extends to graphic representations such as pie charts (Nunes & Bryant, 1996).

The following table contains addition differences that can possibly interfere with student understanding of fractions (Stafyladou and Vosniadou, 2004, p. 505):

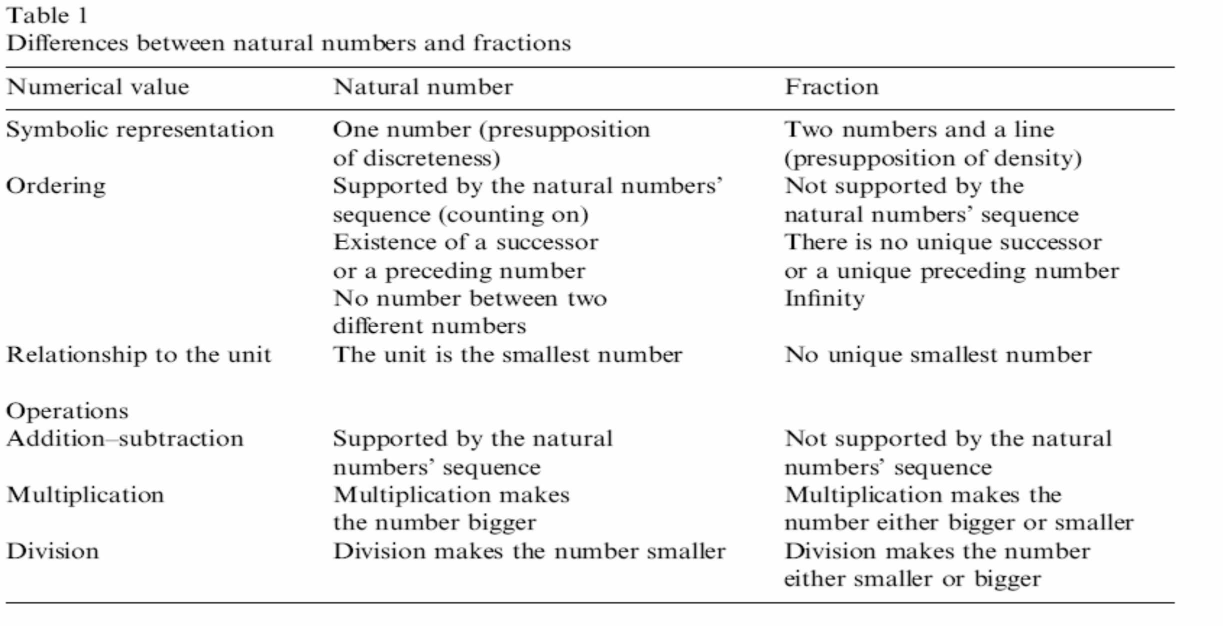


Table 1
Differences between natural numbers and fractions

Numerical value	Natural number	Fraction
Symbolic representation	One number (presupposition of discreteness)	Two numbers and a line (presupposition of density)
Ordering	Supported by the natural numbers' sequence (counting on)	Not supported by the natural numbers' sequence
	Existence of a successor or a preceding number	There is no unique successor or a unique preceding number
	No number between two different numbers	Infinity
Relationship to the unit	The unit is the smallest number	No unique smallest number
Operations		
Addition–subtraction	Supported by the natural numbers' sequence	Not supported by the natural numbers' sequence
Multiplication	Multiplication makes the number bigger	Multiplication makes the number either bigger or smaller
Division	Division makes the number smaller	Division makes the number either smaller or bigger

Focus of Instruction

Researchers have identified several instructional problems linked to student misconceptions.

First, **teachers often fail to account for students' intuitive understanding of rational numbers**. Some research suggests that building on this informal knowledge, students can develop their own computational procedures for solving rational-number problems. Instead, traditional instruction generally does not seek to build on existing understanding, but rather to supplant it with computational algorithms (Kilpatrick, Swafford, & Findell, 2001).

Second, **instruction tends to focus on mathematical procedures for calculating rational numbers without grounding those procedures in any kind of conceptual framework**. Students learn rational-number computation as a series of discrete rules to follow. For example, they learn that to find the common denominator among two fractions, both the numerator and denominator of each fraction are multiplied by the denominator of the other. But they do not understand why fractions must be of the same denomination to be added or subtracted, or that the cross-multiplication process is simply a way of multiplying both fractions by one. Moss and Case (1999) refer to this as a *syntactic* understanding of rational numbers (a series of procedures to be followed) rather than a *semantic* understanding emphasizing meaning.

Third, as described above, certain representations of the unit, or whole, appear to confuse students. Specifically, **it appears that students do not always have a frame of reference for situating rational numbers**. In the case of percentages, the referent may be relatively obvious. (If told that a glass of water is 75% full, most students can deduce that 100% denotes a full glass.) On the other hand, if students are told that the distance from Philadelphia to Pittsburgh is 6/10 of the distance from Philadelphia to Chicago, it is less likely that they understand that the latter distance represents the whole to which the former refers.

In sum, mathematics instruction tends to confuse students' understanding of rational numbers because it fails to build on their preexisting conceptions and relies too heavily on constructs (both notation and computational procedures) that divorce numerical operations from conceptual understanding.

Issues Specific to the Form of the Rational Number

Children have difficulty developing a concept of the size or magnitude of a rational number. With whole numbers, children can more easily grasp the size of a number such as 4. They can see that 4 is 1 more than 3 and 1 less than 5. However, because the counting strategies fail to work in the rational numbers, children have difficulty determining the size of a rational number. In addition, students often fare poorly while doing arithmetic with rational numbers, whose rules differ from those of the whole numbers. Some of the problems students face in coming to understand rational numbers are specific to fractions and some to decimals.

***Fractions*.** Students must come to understand fractions as one entity instead of two numbers that relate to one another. Because the expression of fractions is complex, students tend to view them in relational rather than absolute terms. Given the notation used for common fractions, e.g., 3/4, it is relatively easy to see why this is the case—the notation looks like two numbers rather than one. Lost in this conception, however, is the idea that a rational number is still a number in absolute terms: It can be located along a number line in the same manner as any whole number. While understanding rational numbers in relational terms may be helpful in some respects (such as the understanding of ratio), it may also cause confusion when students are asked to convert one expression to another (Kilpatrick, Swafford, & Findell, 2001). To make sense of these conversions, students must be able to view a rational number as a "fixed point" that can be expressed in multiple ways.

Further, ordering fractions is more complex than ordering whole numbers. Comparing 1/4 and 1/6 conflicts with children's whole-number ideas. Six is greater than 4, but 1/4 is greater than 1/6. With fractions, the more equal parts you partition a unit into, the smaller each part becomes. In contrast, 3/5 is greater than 2/5 because 3 of the same-size parts are greater than 2 of the same-size parts. Being able to order plays an important part in estimating fraction addition and subtraction. Ideally, when a student adds, for example, 1/4 + 1/3, she should be able to reason from her mental images of the symbols that (a) the answer is greater than 1/2, but less than one, and (b) 2/7 is an unreasonable answer because it is less than 1/2.

In two studies by Stafylidou and Vosniadou (2004), the authors examine the development of students' understandings of the size of fractions. In the first study, they look at how students decide what is considered a small fraction and what is considered a large one. They found that children first think that the size of a fraction increases as the numerator increases. This is followed by children thinking that fractions increases in magnitude as the denominator decreases. Next, children believe that fractions increase in size as the numerator comes close to the value of the denominator. Finally, children understand that the size of a fraction grows as the numerator becomes bigger than the size of the denominator.

In the second study, Stafylidou and Vosniadou (2004) examined how children related the relative size of fractions. In other words, they examined how children ordered a group of fractions. They found that children first believed the value of a fraction increases as the numerator or denominator increases. They next believed the opposite that the value increases as the numerator or denominator decreases. Following this, children were able to put fractions in order, but misplaced the number one. Lastly, children were able to put the fractions and the one in order.

Through both studies, the authors were able to come to a conclusion about the development of children's understanding of fractions. At first, children see fractions as two independent numbers. Second, they can only see fractions as part of a whole. Third, children are able to recognize that fractions are actually a relationship between a numerator and a denominator.

Understanding fraction equivalence is not as simple as it may seem, but it is considered to be an extremely important concept in understanding rational numbers. Some children have difficulty noting equivalence from pictures. Imagine a circle partitioned into fourths with one of those fourths partitioned into 3 equal parts. Some children are unable at first to agree that 3/12 equals 1/4 even thought they agree that physically the 2 sections were the same size. Children said that once the lines were drawn in, you could not remove them. In reality, that is just what must be done to understand fraction equivalence from a picture.

Difficulties children have with fraction addition and subtraction come from asking them to operate on fractions before they have a strong conceptual understanding for these new numbers. They have difficulty understanding why common denominators are needed so they revert to whole-number thinking and add numerators and denominators.

Decimals. There are many misconceptions specific to decimal numbers and the differences between the rules that apply to whole numbers and those for decimals:

- As mentioned previously, there is some confusion over which whole-number properties apply to decimals and which do not. Specifically, students tend to overgeneralize features of the whole-number system to decimal fractions and misuse whole-number rules when working with decimals.

- Students have trouble judging the relative magnitude of a decimal number if they have different numbers of digits to the right of the decimal point and treat decimal numbers as whole numbers. Many students assume that more digits to the right of the decimal means a bigger number and fewer digits means a smaller number.

- Other students think that the more numbers there are to the right of the decimal, the smaller the number (Resnick & Nesher, 1983). This would show an overgeneralization of decimal features instead of whole-number features.

- Further, students have trouble working with zero appropriately. Students observe that with whole numbers adding a zero to the right increases the number by a factor of 10 while adding a zero to the left of the original number adds no value. However, students often do not understand that the opposite applies to numbers to the left of the decimal point. Zeros are not viewed as a representation of quantities.

- Unlike addition and subtraction with decimals, multiplication and division notions are not the same as with whole numbers; multiplication does not always make the number bigger, and division does not always make the number smaller.

As mentioned before, the emphasis in instruction must be to develop a rich conceptual understanding of decimals and especially the base-10 structure of the number system as it relates to decimal place values, after making sure the students understand the system for whole numbers. While most students understand that they must “line up the decimal points” when working on a problem, they do not understand why. Students in fifth and sixth grades reportedly believed it was permissible to get different answers to the same problem if it were expressed in different ways, simply because the methods used to solve the problem were different. This shows an overreliance on procedures and a lack of conceptual understanding. Time should be devoted to connecting procedures with the rationale behind them (Hiebert & Wearne, 1986).

What Are Some Strategies to Address Student Misconceptions About Rational Numbers?

As suggested above, most student misconceptions about rational numbers arise from a lack of conceptual understanding: students fail to distinguish between the properties of rational and whole numbers, struggle to identify proportional relationships in context, and tend to seek computational algorithms to find the right answer without understanding the conceptual basis for such algorithms. Given the nature of these misconceptions, it is not surprising that instructional strategies for addressing them focus on (a) building on informal knowledge, (b) linking rational-number concepts to real-world settings, and (c) allowing students to develop their own computational processes based on conceptual understanding. Proportional reasoning and a conceptual understanding of the whole, or unit, are central elements of these instructional approaches.

- **Construct benchmarks based on prior knowledge.** By the time they enter the middle grades, students have developed basic, semantic understanding of the concepts of “one half” and “one quarter.” Using a beaker full of fluid or some other fixed representation of the whole, these basic constructs are used to establish benchmarks that students can use to learn percentages and smaller partitions of the whole. For example, using prior knowledge that one half of one half was one quarter, and that one half was equal to 50%, students were able to identify relations between smaller fractions and percentages (e.g., 1/8 = 12.5%) by working from known benchmarks (Moss & Case, 1999). Following this logic, some research suggests that students should develop a basic understanding of proportion and percent (which are more in line with their prior knowledge) before moving on to fractions and decimals (Moss, 2002).

- **Create real-world examples of the unit, or whole.** For example, a group of fifth- and sixth-grade students used “red light, green light,” a popular schoolyard game, to demonstrate the relation of fractions or decimals to the whole. Students actually played the game for a few minutes, advancing toward the teacher when she called “green light” and stopping when she called “red light.” The teacher then stopped the game and asked the students to figure how best to notate each one’s progress toward the teacher. In this exercise, students were free to use fractions or decimals. In this case, the whole was clearly demonstrated by the goal of the game, with the various parts (fractions) representing students’ progress toward attaining the goal (Bay-Williams & Martinie, 2003). Other approaches use beakers full of water to represent the whole (Moss, 2002; Moss & Case, 1999).

  The construction of real-world examples of the unit is referred to by Kent, Aronsky, and McMonagle (2002) as the “representational context” of the problem, and has been identified as a critical strategy for developing proportional reasoning. The authors introduce recipes as a useful representational context. Most students understand that in a recipe, certain amounts of certain ingredients are combined into a whole of a given size (often measured by the number of servings). It is therefore intuitive that when the size of the whole is increased, the amounts of each ingredient must also increase. If presented with a cookie recipe that makes 24 cookies, and asked how much of each ingredient would be required to make 36 cookies, students are compelled to (a) establish the proportional relationship between 24 and 36, and (b) apply that relationship to each of the ingredients included in the recipe.

  With specific reference to decimals, time offers a useful representational context. Using a stopwatch, students can measure the passage of time in hundredths of a second. The decimals denoting hundredths may not be intuitive to students, but by attempting to track the number of hundredths in a second, they eventually draw a connection between the decimals used by a stopwatch and other rational numbers based on division by 100, such as percentages (Moss, 2003).

- **Assess students' understanding of rational numbers.** Research on students' understanding of fractions suggests that open-ended tasks are best suited for assessing conceptions and misconceptions. Open-ended tasks allow students to display their thinking and reasoning. When followed by probing question, the teacher or researcher can identify the knowledge employed by the student in addressing the task. This goes beyond the normal process of showing work, requiring students to explain, either verbally or in writing, the logic of their responses.

- **Multiple displays of rational-number representations.** Rational numbers can be visualized in several ways, including area models, number lines, fraction blocks or similar three-dimensional objects, as well as numerical and written expressions. Using real-world contexts, students begin with displays of representations that are more intuitive (such as percentages or simple fractions) and work toward those that are less accessible, such as decimals or more complicated fractions (Kilpatrick, Swafford, & Findell, 2001).

- **Use literary or descriptive approaches to introduce proportion.** Using either existing literature or texts created specifically for the task, short stories can be used to create a context for asking and answering questions about proportion. Such texts in effect substitute for the real-world representational contexts described above (Thompson, Austin, & Beckman, 2002).

- **Allow students to develop their own algorithms for solving rational-number problems.** Research suggests that as students develop conceptual understanding of the unit along with proportional reasoning skills, they become increasingly adept at formulating their own algorithms for solving rational-number problems. Weinberg (2002) found that in a "centimeters to miles" problem, the 23% of students who answered the problem correctly employed at least four different computational strategies. Building from students' informal understanding of proportion and percent, Moss and Case constructed a series of games in which students were asked to compare different rational-number representations. For example, students played a card game similar to "War," in which each player lays down a card at the same time. Whoever has the higher-valued card wins the hand and takes both cards. In the modified game, the cards showed different rational-number representations (e.g., .7, 5/8, or 38%). With each hand, students had to figure out which card had the higher value in order to know which student had the better hand. Though students were given no algorithms for making these comparisons, they proved quite adept at developing their own. Post-tests showed that this process also led to greater conceptual understanding of rational numbers and less reliance on memorization of algorithms.

- **Spend time on developing students' understanding of the decimal notation and the naming of decimal numbers.** Teachers claim that less instructional time is required later for operating on decimal numbers if students first understand decimal notation and its roots in the decimal place-value system that we use. Before beginning instruction on decimal numbers, more instruction should be provided on place value as it is used for whole numbers. Problems that require students to work with powers or multiples of 10 give students a flexibility that is useful with whole numbers, and this makes it easier to extend these lessons to decimal numbers. The naming of decimal numbers also needs special attention. The source of confusion is compounded by the use of the "ths" ending (thousandths, hundredths) with decimal numbers.

- **Make sure that students understand the place-value system of decimals.** Students who try to make sense of math become very confused when they are told to add zeros so the numbers are the same size when comparing numbers such as 0.45 and 0.6. This strategy does not develop any sense of number size for decimal numbers. It would be more appropriate to expect students to recognize that another name for 6 tenths is 60 hundredths, which is more than 45 hundredths. Students will think this way if they have been given sufficient time to explore place values, using manipulatives as representations for numbers. Hiebert (1992) discusses research showing that if students do not have a sound understanding of place value when they learn to add and subtract decimal numbers, they make many errors that are difficult to overcome because they are reluctant to relearn how to operate on decimal numbers in a meaningful way (Sowder, 1997).

- **Teach reasonableness and estimation.** Once students are able to understand the value of fractions, decimals, and percents, then they can estimate their value quickly, before going through the steps to solve the problems. In this way, they can determine what the correct answer should be and then determine the reasonableness of their answer. For example, a student needs to be taught to estimate that 7/8 is almost one and that 5/12 is almost 50 percent or 1/2, so they can estimate that the adding the two numbers together will result in an answer between 1 and 1.5. If their answer after computation falls in this range, then they know it is reasonable.

- **Let children use concrete models over extended periods of time.** Cramer and others from the Rational Numbers Project have studied how to teach these concepts for over a decade and developed the following four beliefs: (a) Children's learning about fractions can be optimized through active involvement with multiple concrete models; (b) most children need to use concrete models over extended periods of time to develop mental images needed to think conceptually about fractions; (c) children benefit from opportunities to talk to one another and with their teacher about fraction ideas as they construct their own understandings of fraction as a number; and (d) teaching materials for fractions should focus on the development of conceptual knowledge prior to formal work with symbols and algorithms (Cramer, et al., 1997a, 1997b). However, of the four pedagogical beliefs, the second was determined to be the most important. In order to develop fraction sense, most children need extended periods of time with physical models such as fraction circles, Cuisenaire rods, paper folding, and chips. These models allow students to develop mental images for fractions, and these mental images enable students to understand fraction size. Students can use their understanding of fraction size to operate on fractions in a meaningful way.

- **Make sure that students develop a good "number sense."** Multiple researchers have talked about the importance of developing a conceptual understanding of rational numbers or number sense. This takes time, and often teachers move from the conceptual to the computational too soon, before students really understand what rational numbers mean. Once students start working on procedures and move away from manipulatives and conversations about the meaning of fractions and place values, it is hard to go back and correct a student's number sense. Make sure students have that rich understanding before moving from concept to procedure.

Conclusion

The most common errors in students' understanding of rational numbers arise from two sources: inappropriate application of whole-number concepts to rational-number problems, and reliance on computational processes or algorithms without sufficient understanding of underlying concepts. Focus on concepts, and let the algorithms wait. Let students work over a period of time with multiple manipulatives to visualize the value of rational numbers, and make sure they can explain how and why they reached their answers. Also spend time with estimation and helping students to determine the reasonableness of their answers. Once these essentials are in place, the algorithms will become more than a series of steps without a conceptual framework.

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