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PCK Tools

Symbolic Representation: Student Misconceptions and Strategies for Teaching

The National Council of Teachers of Mathematics (NCTM) notes:

> Many adults equate school algebra with symbol manipulation—solving complicated equations and simplifying algebraic expressions. Indeed, the algebraic symbols and the procedures for working with them are a towering, historic mathematical accomplishment and are critical in mathematical work. But algebra is more than moving symbols around (NCTM, 2000, p. 37).

In one sense, symbolic representation is the *language* of algebra: Equations, formulas, mathematical symbols, and variables are some of the tools used to express mathematical relationships formally. One of the challenges for students is to interpret these symbols correctly. The rules for interpreting and manipulating mathematical symbols are not always in agreement with the way relationships are expressed through the English language. And as we will see, this can cause difficulties for students. Another language-related challenge is one of translation—converting real-world situations into equations, tables, and graphs (and vice versa) is an important way to understand and solve problems, but is not always straightforward.

In another sense, however, symbolic representation is part of the *process* of "doing" algebra: Converting situations and problems into equations, manipulating and solving equations, expressing relationships as functions, etc., are examples of some of the activities one engages in when "doing" algebra. The challenge here is for students to recognize patterns and relationships, to think about them abstractly and generally, and to describe them in mathematical notation.

Research on Symbolic Representation

Importance of Symbolic Representation

The NCTM algebra standard calls for all students, preK-12, to

- understand patterns, relations, and functions;

- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships; and
- analyze change in various contexts (NCTM, 2000, p. 37).

Among these items, the second and third items fall squarely in the domain of symbolic representation. NCTM goes on to suggest that "[mathematical modeling of phenomena is] one of the most powerful uses of mathematics. ... Students at all levels should have opportunities to model a wide variety of phenomena mathematically in ways that are appropriate to their level" (p. 39). Kaput (1999) notes, "the use of algebraic representations … is among the most powerful intellectual tools that our civilization has developed. Without some form of symbolic algebra, there could be no higher mathematics and no quantitative science; hence no technology and modern life as we know them" (p. 134).

Relationship Between Symbolic Representation and Algebra

Symbolic representation is a broad topic, and algebra is even broader. To examine the relationships between them we will use Kaput's (1999) conception of algebra as composed of five major areas:

- generalizing and formalizing patterns (for example, 1 car has 4 wheels; 2 cars have 8 wheels; x cars have 4x wheels);
- rule-based manipulation of symbols and expressions (for example, solving linear equations);
- the study of rules and structures based on computation (for example, even + odd = odd; odd + odd = even);
- the study of functions and relations (for example, examining the relationships between velocity, time, and distance traveled); and
- a modeling language (for example, describing and graphing the motion of physical objects).

Although the use of symbolic notation dominates in element 2, each of the five elements relies on symbolic language as a communicative and representational form. In this sense, symbolic representation lies at the core of algebra. However, for each of these aspects of algebra, the entry-level concepts can also be engaged through ordinary language.

Of these five elements, it has been noted that element 2—involving mostly rule-based manipulation of symbols—is the one most likely to be encountered in the classroom. This represents a narrow view of what algebra is and limits students' opportunities to engage in mathematical thinking (Kaput, 1999; Smith, E., 2003).

Erick Smith (2003) notes that although symbol-based algebra has been dominant for the last 400 years, for the previous 3,000 years algebra was based on ordinary language; that is to say, algebraic ideas were conveyed primarily verbally. The verbal and logical aspects of mathematics are often referred to today as "algebraic reasoning" and are more accessible as an entry point to algebra for younger learners. Smith notes that the first, fourth, and fifth of Kaput's elements of algebra begin with this kind of verbal reasoning and provide a good means of introducing algebra and algebraic thinking in the earliest grades.

Big Ideas in Symbolic Representation

One of the important ideas regarding symbolic representation is that while it is a language, its rules are not the same as those of ordinary verbal languages. As a result, those rules can be confusing to students. Erick Smith (2003) notes that in order to succeed at algebra, one needs to learn the mathematical and logical concepts, but at the same time learn the correct way to express those concepts. As Stacey and MacGregor (1999) note:

> [Algebra] often cannot say what we want it to say. For example, we can represent "y is more than x" as $y > x$, but we cannot represent the statement "y is 4 more than x" in a parallel way. We must make inferences from the unequal situation just described to write such equalities as $y = x + 4$ or $y - 4 = x$. As a second example, the natural way to describe a number pattern like 2, 5, 8, 11, 14, ... is to focus on the repeated addition, perhaps saying, "Start at 2 and keep on adding 3." However, the algebra that most students are first taught cannot express this easy idea in any simple way. To construct the required formula, $y = 3n - 1$, students have to look at the relationship between each number and its position in the sequence. Algebra is a special language with its own conventions. Mathematical ideas often need to be reformulated before they can be represented as algebraic statements (p. 110).

Another big idea in symbolic representation is the concept of abstraction. Mason (1989) describes abstraction as a sort of spiral process. At the lowest level, one begins by manipulating objects (for example, a sequence of numbers). After a while, one "gets a sense of" those objects and begins to be able to articulate rules and properties that describe those objects (for example, certain terms in the sequence are divisible by a number). Having rules and properties to work with instead of the objects themselves is one level of abstraction. When we say, for example, *Nate is 4 years older than Alex,* and represent this as $N = A + 4$, we focus on the relationship between the ages and not on any particular age. Recognizing and describing patterns in this manner is a key aspect of algebraic thinking.

Another key concept in algebra and symbolic notation is the notion of equivalence—what is meant by the = sign. Some students make the conceptual error that = means "do the calculation on the left-hand side and put the answer on the right-hand side." One explanation for why students make this error is that they generalize from the arithmetic and calculation problems that are so prevalent in elementary school. Here, the equal sign is often used to signal “get the answer” with students rarely seeing it used to express a relationship. This (mis)understanding of the equals sign can be thought of as a result of instruction. Given this starting point, it may be very difficult for these students to make sense of the rules for manipulating expressions and solving equations.

In looking at the = sign, one researcher suggests five different possible uses, as illustrated in the following expressions:

1. A = LW
2. 40 = 5 * x
3. sin x = cos x * tan x
4. 1 = N * 1/N
5. y = kx

“We usually call (1) a formula, (2) an equation to solve, (3) [the statement of] an identity, (4) [the statement of] a property, and (5) an equation of a function of direct variation (not to be solved.)” (Zalman Usiskin, as cited by Chazan & Yerushalmy, 2003, p. 125). In each of these statements, the = expresses a relationship, but the nature of those relationships is different, and the implications for interpreting the variables involved is also different, and in only one case is the equation intended to be solved. Chazan & Yerushalmy suggest that students that are relying on rules and procedures to make sense of mathematical notation are more likely to be confused in applying those rules. They suggest that for these students it may be difficult to know when a rule applies and how to apply it.

Misconceptions and Errors

The Reversal Error

- An often-cited representation error is known as the “reversal error.” The error has been described using this situation:

  There are 6 times as many students as professors. Express this relationship using S as the number of students and P as the number of professors.

In many studies, over half the respondents mistakenly offer the relationship 6S = P (Clement, Lockhead, & Monk, 1981; Clement, 1982). There are at least two major theories offered to explain this common error.

One theory suggests that students translate the key elements of a problem—“six,” “students,” “professors”—to an equation in the same order those elements appear in the word problem. This incorrect method (sometime called “syntactic translation” or “word-order translation”), though important, is believed to be less common than another method where students may correctly understand what the problem is saying (i.e., there are more students than professors), but then mistranslate that understanding into mathematical symbols (Clement, Lockhead, & Monk, 1981; MacGregor & Stacey, 1993).

In this second method, interviewed test subjects report a typical thought process going something like this:

> There are six students for every professor. Let 6S stand for six students, and let P stand for one professor. Then 6S = 1P.

Two errors are worth noting in this approach. Here, S becomes a label for the number 6, rather than standing for the number of students. Even in experimental situations where subjects are explicitly told that S and P should stand for the *number of* students and professors, respectively, many will revert to the interpretation 6S stands for six students. The second major error is that in this representation, the = sign is used to denote a correspondence between unequal quantities (i.e., six students go with each professor) rather than a true equality (Clement, 1982).

Interviews of subjects with correct answers suggest a common correct method that respects the meaning of the = sign. Students with correct representations recognize that there are more students than professors, and thus S is larger than P. But then these students take the additional step to consider what is needed to make S and P equal, namely to multiply P by 6, and therefore S = 6P (Clement, 1982).

There are several additional points worth noting about the reversal error:

1. It is very common. In some studies, error rates were between 40% and 90%.

2. It appears with addition problems as well as multiplication. For example, in the problem, *Express the relationship: Joe has 5 more marbles than Bob*, students will mistakenly put the number with the larger quantity. Thus, J + 5 = B.

3. Students make the same sort of reversal errors describing tabular or pictorial data as they do with word problems. For example, when a graph or table of data indicates an unequal relationship between two quantities, students will make errors as described above in attempting to express the relationship as an equation, yet which express the relationship as an equation without maintaining the concept of equality, namely that both sides of the equation must be equivalent.

4. The error is very robust. Even when directly questioned, students will tend to stick with their incorrect equations. In cases where students, upon questioning, carefully reason their way to a correct solution, they will often when forced to choose, revert to the incorrect solution (Clement, 1982).

Parsing Errors

A broad category of error arises from students misreading or misunderstanding algebraic notation. These errors, sometime referred to as "parsing errors," include but are not limited to the following:

- Students may read 5x not as 5 times x, but instead as one would read 5½ (5 plus one half), or as one would read 53 (5 in the tens place and 3 in the ones place). This is a matter of not realizing the implied multiplication of juxtaposing 5 and x (Stacey & MacGregor, 1999).

- Students may misinterpret 2 + 3x as being equal to 5x. This error can be seen as failing to heed the order of operations (multiplication before addition), but also could be seen as an example of reading from left to right as in ordinary English (Tall & Thomas, 1991).

- When faced with multiple instances of a variable in an equation, for instance 3x + 2x = 30, students might respond, "Well, if this x were 4, then this x would be 9." Such an error might indicate that the student was not aware of the convention that a given letter would have the same value each time it occurs in a given equation (Carpenter, Franke, & Levi, 2003; Sleeman, 1984).

- In a related misconception, "when students learn that the same variable must be replaced by the same number, they often overgeneralize to assume that when the variables are different, they cannot be replaced by the same number. For example, they assume that 4 cannot be substituted for both w and z in the number sentence w + z = 8" (Carpenter, Franke, & Levi, 2003, p. 75).

- Students interpret symbols as actions rather than expressions of relationships. For example, the following problem can be interpreted in a number of incorrect ways:

    Given, a = 28 + b, pick the correct relationship:

    1) a > b
    2) a < b
    3) a = 28
    4) can't tell

  About one-quarter of students at all levels made incorrect choices, and in several schools, the percent wrong was far greater. Some students thought that since the letters stood for unknown numbers, they could not tell which was greater: "They could be anything." Some thought that b was greater because it had 28 added to it, whereas a had nothing added to it. Some thought that a equaled 28 because the equation said "a equals 28, then add b." (Stacey & MacGregor, 1999, p. 112).

Here, response 3 may be due to a misconception acquired through previous experience with arithmetic where students believe that the = sign means “the answer comes next.”

- In the problem 8 + 4 = __ + 5, some students offered the answer 12. This may be seen as the same type of error as response (3) immediately above, namely that the = sign means “the answer comes next,” rather than as a statement of a relationship between equal quantities. In a similar vein, the = may be seen as a command to “do a calculation.” Another interpretation of this type of error suggests that students treat problems like this as sentences to be read from left to right namely, 8 plus 4 equals 12, then add 5 (Stacey & MacGregor, 1999).

Conceptual Errors

The following errors are not due to misinterpretations of language, but may be seen as misconceptions about variables and functional relationships.

- Stacey and MacGregor (1999) gave students the following problem, *David is 10 cm taller than Con. Con is h cm tall. What can you write for David's height?* They found some students used letter substitution a=1, b=c, c=3, etc. to determine that h=8 and David's height was therefore 18.

- Stacey and MacGregor (1999) also report:

  > The belief that any letter alone stands for 1 was another obstacle for older students. Students explained that “by itself the letter is one thing, 1” and that “x is just like 1, like having one number.” One likely cause of this belief is a misunderstanding of what teachers mean when they say “x without a coefficient means 1x.” The student gets a vague message that the letter x by itself is something to do with 1 (p. 112).

- Sleeman (1984) suggests that students learning new algebraic concepts may overgeneralize and misapply their new knowledge. He offers the example that when students learn that $(A * B)^C = A^C * B^C$, they may mistakenly conclude that this applies as well to $(A + B)^C = A^C + B^C$.

Conceptual Obstacles

Three additional misconceptions are reported by Tall and Thomas (1991) that may not surface outwardly as errors, but serve as obstacles to students’ developing a flexible understanding of algebraic notation.

- The first, termed the “expected answer obstacle,” is the belief that, similar to the expression 2 + 3, which can be evaluated, expressions such as 2 + 3x must also be evaluated, even if the value of x is not able to be determined.

- A related obstacle they call the “lack of closure obstacle.” This occurs when students encounter expressions such as the one above. They may mistakenly believe that the expression needs to be evaluated before they can proceed to use it.

- A third, and again related, obstacle is termed the "process-product obstacle"

  > ... caused by the fact that an algebraic expression such as 2 + 3a represents both the process by which the computation is carried out and also the product of that process. To a child who thinks only in terms of process [for example], the symbols 3(a + b) and 3a + 3b . . . are quite different, because the first requires the addition of a and b before multiplication of the result by 3, but the second requires each of a and b to be multiplied by 3 and then the results added (p. 126).

Conclusion

While some of the errors presented here revolve around misinterpretations of notation, many of the significant misconceptions and errors seem to be related to students misapplying arithmetic thinking to algebraic relationships. In many cases, students see algebraic expressions as computations to perform, rather than as expressions of relationships between quantities. In many ways, this can be seen as a holdover of mathematical instruction in the elementary grades.

To help lead students past the arithmetic, many suggest that teachers work with students to develop their observations of patterns and relationships, first in words, and later moving towards standard notational representations. As students make sense of simple relationships and practice verbalizing those relationships, they gain experience with the concept of abstraction from the earliest grades, which prepares them for the increasingly rigorous use of symbolic notation in the later grades

So many of the errors described above can be thought of as a direct result of the way that algebraic concepts are traditionally taught in school. Students are often asked to perform actions—simplify, evaluate, solve, etc.—rather than actually use algebraic concepts and symbols to represent and solve real or relevant situations. Because they are not exposed to the process of algebraic thinking and reasoning, the rules for manipulating and interpreting symbolic expressions have little meaning and are simply rules to memorize, or "rules without reasons." Instead, as suggested by the NCTM standards, students need exposure to the process of beginning with a situation, representing and generalizing the mathematical relationships with symbols, and using equations to model the situation.

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