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Consortium for Policy Research in Education

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PCK Tools

Variables and Functions: Student Misconceptions and Strategies for Teaching

French (2002) points out that many students view algebra as a difficult subject, lacking in both meaning and purpose. Many only learn algebra well enough to complete homework and pass tests. Their experience with school algebra fails to give them a clear view of what algebra is, and they develop a narrow image of the subject. The National Council of Teachers of Mathematics (NCTM) suggests that one way of trying to find out what makes algebra difficult is to identify the kinds of errors students commonly make in algebra and then to investigate the reasons for these errors (Booth, 1988).

Herscovics and Linchevski (1994) argue that the many of the difficulties that students experience with algebra stem from the pace with which it is covered and the formal approach often used in its teaching. They also point out that many teachers and textbook authors fail to address serious cognitive difficulties involved in the learning of algebra. Hence, a vast majority of students does not construct meaning for the abstract symbolism and are restricted to performing meaningless operations on symbols they do not understand.

Algebraic thinking should start by fostering the ability to recognize, describe, and extend patterns. Once the patterns are recognized, the use of a variable helps representing that pattern in an algebraic form. This development can be further extended as students learn to represent and analyze mathematical situations and structures. Students should then learn to use relations and functions to analyze change in a variety of contexts. (Hatfield, et al., 2005). After beginning with some basic definitions, we will discuss the ways students interpret the concept of variable, some difficulties they have with functions, and recommendations for instruction.

Definitions

Throughout the following discussion, there is a good deal of terminology dealing with variables and functions. Therefore, this section is to be used as a short glossary[1].

Expression. An expression is a mathematical statement involving numbers, letters, and/or operations. An expression can be a single entity or a long chain of characters. Students are

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[1] These definition are adapted from the Wolfram MathWorld reference at http://mathworld.wolfram.com

usually asked to *evaluate* expressions. This typically means that the expression is to be simplified—where numbers and/or variables are combined, when possible.

Equation. An equation is a mathematical sentence in which two expressions are set to be equal, signified by an equals sign. Students are usually asked to *substitute* values into an equation, or to *solve* or *graph* them.

Variable. A variable is a symbol, often a letter, which can represent a number. Variables can also be used in an expression, function, etc. Students are usually asked to *solve* for a variable, or *substitute* a value in for a variable and *evaluate*.

Ordered Pair. Also called a *coordinate pair,* or *Cartesian product*, an ordered pair is a set of two numbers, (x, y), where the first number tends to be associated with an input variable, and the second with an output variable. Order is important here. The words coordinate pair and Cartesian product get their name from the graphical representation of this ordered pair on a two-dimensional space with a vertical and horizontal axis, called the *Cartesian Plane*. This is the mathematical term for the two-dimensional graphs we produce using an x-axis and a y-axis.

Relation. A relation is any set of ordered pairs, (x, y), where each x-value is related to a y-value by some rule. This rule can be written in a formula, or can simply be a list of points. This term is important because it is the basis for a function, which is a special type of relation.

Function. A function is a relation that uniquely associates one set (x-values), with another set (y-values). This too can be generalized in a formula, or simply be a list of points. With a function, each x-value can only relate to one y-value; y-values, on the other hand, can be associated with an infinite number of x-values. A graphical way to show whether a relation is a function is with the *vertical line test*. If you can draw no vertical line (remember that vertical lines represent the x-values) that passes through more than one point, then the relation is a function.

Domain and Range. The domain, also called the *pre-image* by mathematicians, is the set of values that define the input of a function. The range, also called the image, is the set of output values from a function based on the input, or domain. The domain and range of a function needs to be stated, as it is just as important as the formula. The person working with a function needs to know what values are going into it, and consequently what values are coming out of it. With textbooks, unless otherwise stated, it is generally assumed that the domain and range is always the set of real numbers. In addition, there are two qualifications that go with domains. First, some functions cannot accept all the real numbers. For example, the square-root function cannot take in negative real numbers. Therefore, the domain is limited by the function itself. Second, the user can also limit the domain to a set of numbers of some interest. Maybe, one would want to look at the square-root function with x-values from one to nine. Thus, the range is then based on the input chosen by the user and/or set by the function.

Function Definition. Although not typically written in secondary-school textbooks, a properly written function includes function notation and the user-defined domain and range:

$$f : \{\text{natural numbers}\} \rightarrow \{\text{natural numbers}\}$$
$$f(x) = 6x + 4$$

The first line indicates the possible inputs and outputs. Since there are no restrictions by the function, the domain is the natural numbers (1, 2, 3,…). The range is a little more restricted. Since the lowest natural number is one, the lowest number in the range ($f(1)=6(1) + 4$) is 10. So, the range here is the set of natural numbers greater than or equal to 10. The f(x) notation has some interesting properties. Functions can be treated as variables and combined, as in $(f(x))^2$, $f(x) + 2g(x)$, or $3f(x) / 7g(x)$. Functions can also be composed, $f(g(x))$. *Composition* is when one function is substituted into another. Last, functions can be inverted. *Inverse functions* are denoted by the $f^{-1}(x)$ symbol. An inverse function switches the domain and range around; it can be thought of as "undoing" a function, or doing the opposite—addition to subtract and multiplication to division are examples. However, not all functions are invertible. For example, a quadratic function, $f(x) = x^2$, takes in all values, but only outputs positive numbers. The inverse is the square root. The range of the quadratic (no negatives) becomes the domain of the inverse, but the square-root function only outputs positive numbers. So, the range here does not match the input of the original quadratic function, all real numbers.

Types of functions. A function can be either continuous or discontinuous. Although not a mathematical definition, one can think of a function defined by a smooth curve with no holes as continuous, and anything else as discontinuous, like function identified as a set of points. Most functions that students work with are continuous. A *linear* function is one of these. Linear functions are graphed as straight lines, and have equations with no exponents (i.e., $y = 2x – 2$, $f(x) = 3$). A *quadratic* function has a curve with one bend when graphed—in the shape of a *parabola*—and has an equation where the highest power is two (i.e., $y = 2x^2 + 3x – 1$, or $f(x) = -4x^2$). A *cubic* function has a curve with at most two bends, and has an equation where the highest power is three. All of these are considered *polynomial* functions, as the equations are made up of polynomials. Nonpolynomial functions include the sine, square-root, and logarithm function, to name a few. A *piecewise* function can be discontinuous or continuous, and occurs when a function is defined by different rules over a number of intervals (Figures 4a and 5 are examples of this). A piecewise function could be a parabola from negative infinity to -2 on the x-axis, a horizontal line from -2 to 5, and a sine graph from 5 to positive infinity.

The Concept of Variables

French (2002), in describing a study by Küchemann (1981), observes that students give different interpretations to letters in algebra and categorizes them as follows, many of which are misconceptions (though can be used to successfully find solutions to given tasks):

a) *Letter evaluated*: Given a simple equation, usually with only one operation, a student is able to find the value of a variable without algebraic manipulation. Consider the equation $a + 5 = 8$. Since $3 + 5 = 8$ is so familiar, the letter a is immediately assigned with the value 3.

b) *Letter not used*: When a student evaluates an expression, they may work with the numbers and not use the variables. For example, to evaluate $a + b + 2$, given that $a + b = 35$, the focus is on adding 2 to $(a + b)$ and thus the letters can be ignored. As was the case in the previous category, here students have some idea that the letters stand for some numbers, but they do not realize that the letters can be operated on in a way other than substitution.

c) *Letter as object*: Students misinterpret letter as an abbreviation for an object. For example, b may stand for bananas and c for carrots rather than the number of bananas or the number of carrots.

d) *Letter as specific unknown*: Students assume that the variable can only correspond to a single number. Though this interpretation is a valid one in the context of equation solving, conflict may arise while solving equations with more than one solution. In addition, students may not realize that different letters can be used to represent the same equation. They may also fail to see that the two equations $7x + 4 = 25$ and $7y + 4 = 25$ have identical solutions.

e) *Letter as generalized number*: Students realize that a letter can take more than one value. This idea is closer to the concept of variable, where the letter can vary across a whole range of values. When students faced with a challenge to describe c with reference to the given conditions $c + d = 10$ and $c<d$, the most common response was to assign a single value to c, usually 4. However, some other students gave a list of values (for example, …-2, -1, 0, 1, 2, 3 ….) indicating the notion of the letter as a generalized number. The response $c < 5$ suggests a greater understanding of the idea of variable.

French (2002) asserts that the barrier created by these interpretations, especially by *letter as object* needs to be addressed in the early stages of learning. Furthermore, the idea of *letter as number*, whether it is unknown or variable, should be constantly reinforced.

Bell, Costello and Küchemann (1983) studied the APU Primary Survey (1980) conducted in London, and observed that the questions that can be answered successfully by the interpretations and strategies of categories a, b, and c, mentioned above, are accessible to a significantly high proportion of secondary school students. They also suggest that the correct answer to the question, *Which is larger, 2n or n + 2?,* demands the notion of letter as a variable and only 6% of secondary school students could respond correctly that *the answer depends on whether n is greater or less than 2*.

The Function Concept

Markovits, Elyon, and Bruckheimer (1988) observe that students have difficulty with the terms ordered pair, domain, and range, and that this can lead to other difficulties related to graphical and algebraic forms of functions. Many students demonstrate difficulties in

a) *Locating domains and ranges on the axes in the graphical representation.* A related difficulty is the double role of points on the axes. Points with co-ordinates (x, 0) or (0, y) can represent ordered pairs when the function intersects one of the axes. At the same time, they can be seen as values on the axis that are not part of an ordered pair. Finding the domain and range blurs this distinction.

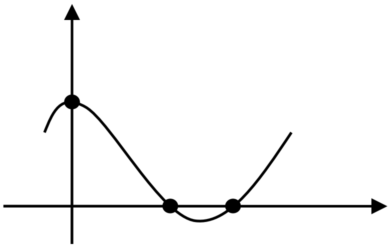


Figure 1

b) *Identifying domains, ranges, and ordered pairs of functions given in algebraic form.* For example, consider the following function and the related questions. Few students answer all parts correctly:

f: {natural numbers} → {natural numbers}
f(x) = 6x + 4

Which of the numbers -2, 4, 10, 16, 20.5 can be
(i) a pre-image under f ?
(ii) an image under f ?

Which of the following ordered pairs is part of f?
(-1, -2), (0, 4), (6, 4), (6, 10), (5, 34)

c) *Distinguishing between the domain and the set of all possible inputs.* Students consider these to be identical. When the domain is a subset of all possible inputs, many students have difficulty in understanding the given graph as a function, even when it is so. Such students do not consider the graph in Figure 2a as representing a

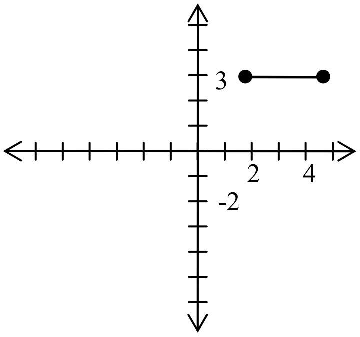


Figure 2a

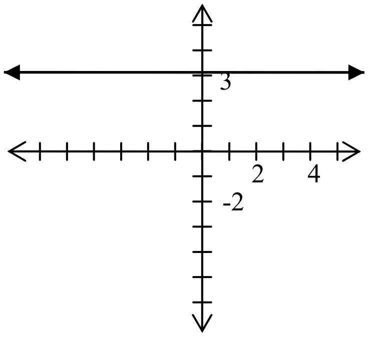


Figure 2b

function with a truncated domain $\{x : 1 < x < 4\}$ and range $\{y : y = 3\}$, while Figure 2b is considered a function because the domain matches contains all possible inputs.

d) *Taking domain and range into consideration*. Most of the students draw the graph of the function;

f : {natural numbers} → *{natural numbers}*

f(x) = 3

as shown in Figure 2b above. This clearly shows that the students ignored the domain of the function and drew the graph of the function as a straight line. Since the natural numbers do not include the space in between 1, 2, 3, etc., the graph should have be a collection of points *not* connected by a line.

Many students often have the misconception that every function is a linear function. Such students think that there can be only one function that contains the points **A** and **B** in Figure 3a and represent it graphically as a straight line, Figure 3b, through **A** and **B.**

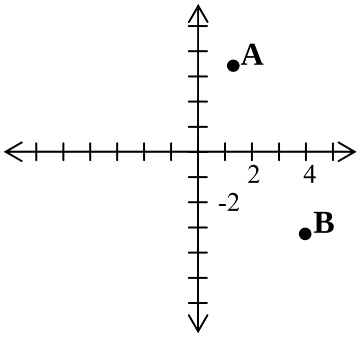


Figure 3a

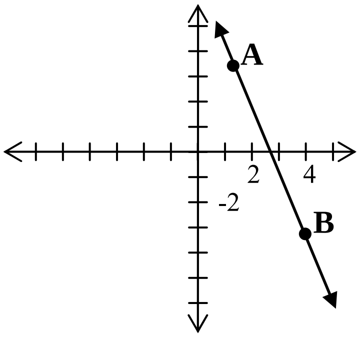


Figure 3b

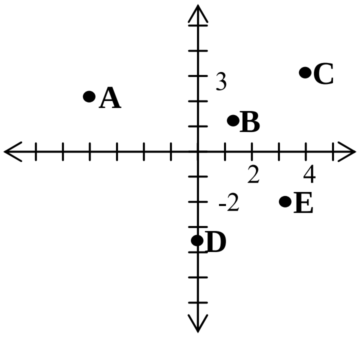


Figure 3c

They may also think that there cannot exist a function that has the coordinates of the points **A**, **B**, **C**, **D**, and **E, as** in Figure 3c. Markovits, Elyon, and Bruckheimer (1988) suggest that including nonlinear functions throughout the chapter dealing with linear function may help students overcome this difficulty. Teachers could illustrate a nonlinear function, including its equation, which contains the points **A**, **B**, **C**, **D**, and **E** (Figure 3c) in its graphical representation. Students should then be given time to plug the points into the equation to prove that the function truly passes through each point exactly.

Markovits, Elyon, and Bruckheimer (1998) observe that students often have difficulties with certain types of functions:

a) *The constant function:* Students usually do not understand that in a constant function, all the x-values have the same y-value. This may be caused by the lack of an independent (x) variable in the equation.

Question For the function *g*, g : {real numbers} → {real numbers} $g(x) = -7$. Complete the following. $g(4) =$ _____ $g(-7) =$ _____ $g(0) =$ _____ $g(3.5) =$ _____	*Answers* $g(4) = -4$ $g(-7) = 7$ $g(0) = 0$ $g(3.5) = -3.5$

b) *Functions represented by disconnected graph:* Some students view the relations shown by Figure 4a and Figure 4b as nonfunctions because of the discontinuity in the graph.

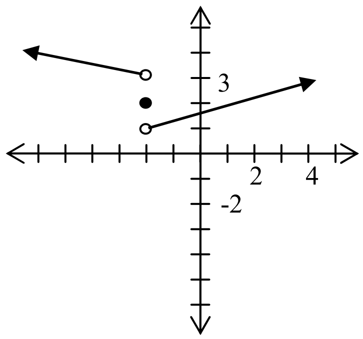


Figure 4a

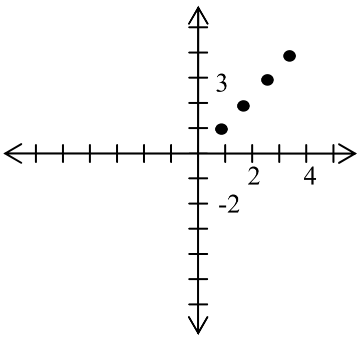


Figure 4b

c) *Functions defined piecewise:* Some students fail to see that the two rules of correspondence in Figure 5 refer to two disjoint parts of the domain. They assume that every domain has two ranges. and so the relation is not a function.

Figure 5

Is the following relation a function?

$$k : \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix} \rightarrow \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix}$$

$$k(x) = \begin{cases} x + 2 & x < 5 \\ x - 3 & x \geq 5 \end{cases}$$

d) *Difficulties caused by technical manipulations:* Some students demonstrate difficulty in solving problems involving technical complexity. Fractions and negative numbers tend to make tasks more challenging for youth. Such students may find it difficult to answer the question in Figure 6. They may face greater difficulty in finding domains than in finding ranges.

Figure 6

Given the function h

$$h:\begin{Bmatrix}\text{real}\\ \text{numbers}\end{Bmatrix}\to\begin{Bmatrix}\text{real}\\ \text{numbers}\end{Bmatrix}$$

$$h(x) = -\frac{2}{3}x - \frac{3}{4}$$

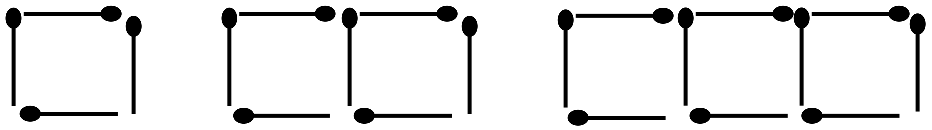
Find

$h(3) = \square$ $h(½) = \square$ $h(\square) = -3$

Implications for Teaching

Stacey and MacGregor (2000), point out that mathematics educators have come to regard *working with generality* as one characteristic of algebra. Consequently, recognition and description of general rules for patterns are seen as part of algebra in recent curriculum documents for schools, and these documents recommend that the first use of algebraic letters should be for expressing the description of these patterns and relationships.

Figure 7



Adapted from Stacy and MacGregor (2000)

For example, the number of matches required to build a row of squares (Figure 7) can be described by the rule, t*he number of matches you need is one more than three times the number of squares.* This can be abbreviated to, $y = 3x + 1$. This approach is a clear break with the traditional approach where the introduction of algebraic notation was based on the use of letters that stood for specific but unknown numbers. However, Stacey and

MacGregor argue that there is little evidence in published studies that might support the shift from the traditional approach to the new one. In their opinion, students may observe a variety of patterns, though only a few of them are useful in describing a functional relationship.

They provide the following question (Figure 8) and students' responses to support their argument.

Figure 8

Look at the numbers in this table and answer the questions.

x	0	1	2	3	4	5	6
y	2	5	8	11	14	17	

(i) When x is 10 what is y?

(ii) When x is 100 what is y?

(iii) Use algebra to write a rule connecting x and y.

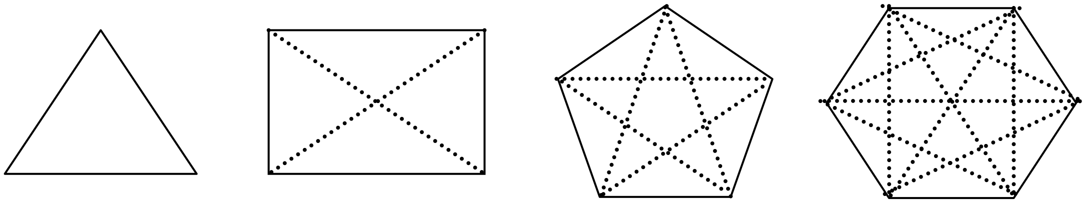
Many students easily see the recurrence relations as, *x goes up by one and y goes up by three* or *an increase of 1 in x causes an increase of 3 in y*, but they fail to see the functional relationship $y = 3x + 2$ that connects the two variables. Therefore, teachers should begin with this approach and then work to make the connection of seeing the functional relationship.

Dettori, Garuti, and Lemut (2000) argue that students should not be limited to performing algebraic activities, but they should be helped to acquire algebraic language as a working and expressive tool. In their opinion, learning algebra after arithmetic means developing a different way of thinking. They also argue that algebra shows its power best when applied to problem-solving, especially at the middle-school level, where students are still too young to appreciate mathematical formalism for its elegance and conceptual power. A good way to introduce algebra is through problem-solving of increasing complexity.

French (2002) suggests that abstraction can act as a considerable barrier to many learners and thus "algebra needs to be seen as a way of solving problems that have significance for the learner." (p.17) According to him, it is difficult to find real-world examples in the early stages of learning algebra. Hence, he holds that "solving a numerical or geometrical problem or explaining a surprising number property can be just as motivating and meaningful as applying mathematical ideas to a financial or travel problem." (p.18)

Consider the example of finding a relationship between the number of diagonals and the number of sides of a polygon. French (2002) puts forward the steps through which his students reached the stage of deriving a formula:

Figure 9



No. of sides	3	4	5	6
No. of diagonals	0	2	5	9

Adapted from French (2002)

1. Teacher presented a task of determining the number of diagonals in different polygons.
2. Students drew their polygons with the diagonals and tabulated their results as in Figure 9.
3. Teacher posed a challenge of determining the number of diagonals for a 20-sided polygon.
4. Students realized the difficulty in finding the answer by drawing the polygon and its diagonals or by following the pattern they noticed in the number of diagonals.
5. With the help of the teacher, students focus on the hexagon and notice that there are 3 diagonals from each of the 6 vertices. Therefore the number of diagonals is $(6 \times 3) \div 2 = 9$. Division by 2 is required because $6 \times 3 = 18$ counts each diagonal twice.
6. Through guided discussion, students notice that for every polygon, the number of diagonals drawn from each vertex is 3 less than the number of vertices (or number of sides). Thus,

$$\text{Number of diagonals of an n-sided polygon} = \frac{1}{2}n(n-3)$$

$$\text{For a 20-sided polygon, the number of diagonals} = \frac{1}{2} \times 20 \times (20-3) = 170$$

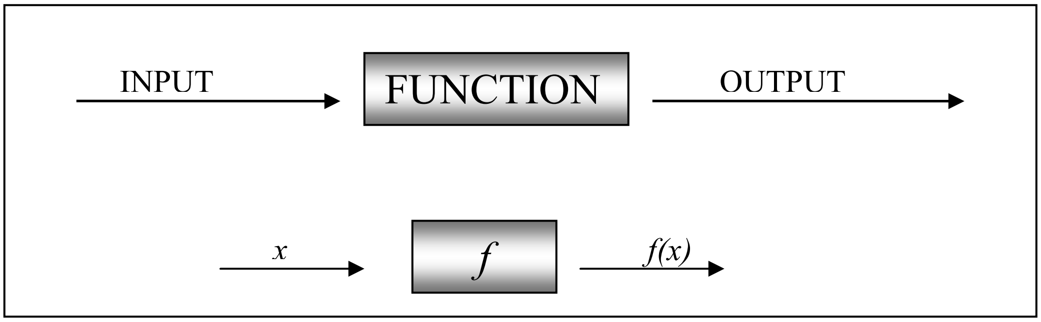
French argues that “the polygons provided a meaningful context that was understood by the students and the power of the algebraic result was evident, because they could now work out the number of diagonals for any polygon.” (p. 19) In his opinion, providing meaning and purpose is essential from the very first introduction to the use of letters, and students should see algebra as making some tasks simpler, rather than complicating simple problems that can be solved easily in other ways.

In studying ways to integrate the study of functions throughout the K-12 curriculum, Smith (2003) observed that most approaches to functions took what may be termed a *relational* approach. That is to say, in the various definitions of functions, elements of one set are *related* to elements of another set, or outputs are *related* to inputs, or y values are *related* to x values through some rule. He suggests a different approach where variations in one variable are examined side by side with examination of changes in another. For example, rather than examine the relation $y = 3x + 1$, one might study the following set of values:

x	1	2	3	4
y	4	7	10	13

Here, as x varies by 1 at each step, y varies by 3. The difference here is that rather than looking at how individual x-values relate to individual y-values, we look at how the series of x-values varies in parallel with the series of y-values. He notes that this is often how functions are examined in science classes where data or other measurements, especially in studies of motion, are collected and compared. Smith argues that this *variational* approach to studying functions may be more intuitive for students. He suggests that by studying patterns in changing numbers that students may have an easier time grasping the concept of function, and that this approach provides a good method for introducing the concept of function at earlier grades.

Figure 10



French (2002) suggests that the idea of a mathematical function as an input-output device (Figure 10) can be introduced at an early age and students can be asked to determine the function for the given input-output series. As a next step, the notation for function can be introduced, whereby an input of x gives an output of $f(x)$. The next stage in developing the function concept is describing the function with an algebraic expression. For instance, the function *square and add one* can be described algebraically as $f(x) = x^2 + 1$. At an elementary level, this provides a simple way of indicating the value of a function for a particular value of the variable—for $f(x) = x^2 + 1$, $f(0) = 1, f(5) = 26$, etc.

Demana and Leitzel (1988) contend that before children encounter formal courses in algebra, basic concepts in algebra should be introduced through numerical computation and problem solving. To be ready for evaluating $7 + 3x^2$ when $x = 0.5$, students should do analogous types of numerical computation in pre-algebra. Evaluating $7 + 0.5^2$ correctly also helps students understand the *order of operations* (parenthesis, exponents, multiplication, division, addition, and last subtraction). The correct use of parenthesis is another important feature of computation that is essential in algebra.

Saunders and DeBlassio (1988) suggest that when students try to visualize a function, they should see its graph rather than a table of ordered pairs. In order to emphasize the function-graph relationship, computers can be of help. Computers can be used to figure out the values for a table and to plot the points, thereby "the students can concentrate on what happens to a function when change are made, like $y = x^2$, $y = x^2 + 2$, and $y = x^2 - 2$." (p.155) House (1988) also asserts that computer software and graphing packages provide the teacher with dynamic means of demonstrating and exploring important concepts related to functions and their graphs, which the traditional chalkboard or overhead projectors cannot do as easily.

Friedlander and Tabach (2001) promote the use of more than one representation from the very beginning of learning algebra and argue that the use of verbal, numerical, graphical, and algebraic representations has the potential of making algebra learning meaningful and effective. The following table (Table 1) shows a consolidated form of their analysis of the four representations:

Table 1. Multiple Representations Used in Algebra

Representation	**Major Use**	**Limitations**
Verbal	• Useful in posing a problem, interpreting the final results, and understanding the natural context of the problem • It emphasizes the connection between mathematics and other domains of academics and everyday life	• The use of verbal language can be ambiguous and elicit irrelevant or misleading associations • It is less universal • Its dependence on personal style can be an obstacle in mathematical communication
Numerical	• Convenient and effective bridge to algebra • Important in acquiring a first understanding of a problem and in investigating particular cases	• Lacks generality • Some important aspects or solutions of a problem may be missed

Representation	**Major Use**	**Limitations**
Graphical	• Provides a clear picture of a real valued function of a real variable • Graphs are intuitive and particularly appealing to students who like visual approach	• Lacks the required accuracy • Influenced by external factors such as scaling • Usually presents only a section of the problems domain or range
Algebraic	• General, concise and effective in the presentation of patterns and mathematical models • Useful in justifying or proving general statements	• Its exclusive use may blur or obstruct the mathematical meaning or nature of the represented objects and cause difficulties in some students, interpretation of their results

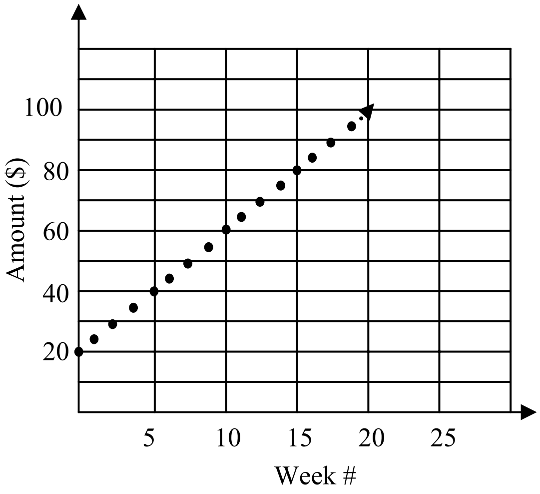
They give an example of a problem situation that can be presented using four different representations, where the students have to investigate the changes in the weekly savings of four children. The first child's savings were presented in a tabular form, the second child's savings in a verbal form, the third in graphical form, and the fourth in algebraic form:

1. The table shows how much money Dina had saved at the end of each week. (The table continues in the same way for the rest of the year.)

Week	1	2	3	4	5	6	7	8	9	…
Amount($)	7	14	21	28	35	42	49	56	63	…

2. David kept his savings at $300 throughout the year.

3. The graph describes Michael's savings at the end of each of the weeks and continues in the same way for the rest of the year.



4. Richard’s savings can be described by the expression $500 - 5x$ where x stands for the number of weeks.

The presentation of the problem situation should be followed by posing investigative and reflective questions. For example, *describe in your own words how the savings of each child changes throughout the year* and *compare the savings of two or more children* are questions that help students get acquainted with the initial representation, as well as switch between representations. A challenge to comment on their own or others work, or to design their own question is a reflective task that may lead students to evaluate their own or others’ actions. Friedlander and Tabach assert that “suitable problem posing and questioning—and systematic encouragement of students’ experimentation with various representations—can increase the awareness of and the ability to use various representations in the solution of a problem” (p.184).

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